Geometry's Future

Conference Proceedings
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Preface

Euclid's *Elements*, written over two thousand years ago, is a stunning accomplishment. Taken together with the insights of J. Bolyai and N. Lobachevsky, Euclid's work, its extensions and analysis of its limitations, have formed the core of what laymen and typical mathematicians see of geometry in high school and in our colleges and universities.

Yet that small group of mathematicians who think of themselves as geometers see their subject in a much broader, deeper, and richer context. Geometers know that, in addition to the many areas that have blossomed within geometry, geometric insights have often provided the germ for conceptual developments in other branches of mathematics, which have now strayed far from their geometric roots. Furthermore, professionals outside of mathematics (e.g., physicists, computer scientists, biologists, engineers, etc.) are finding that geometric ideas have widespread use not only in the theory of their subjects, but also in implementing new technologies that are growing out of these fields. Examples include the use by biologists of knots to study the coiling of complex molecules and the use by computer scientists and engineers of many parts of geometry to develop computer vision and robotic systems. Many concerned geometers have been pained to watch these exciting developments juxtaposed with an unchanging curriculum for geometry in our schools and colleges.

With this as a backdrop, I proposed to COMAP (Consortium for Mathematics and Its Applications) the convening of a small group of geometers to study what could be done to revitalize geometry in our colleges, and what effects this might have on the teaching of geometry in general. It was proposed that special attention be given to a discussion of the survey course in geometry taught at many colleges, a course which often attracts many prospective high school teachers. With the timely and generous support of the Alfred P. Sloan Foundation, twelve geometers were invited to participate in the brain-storming session. Those able to attend, together with several guests, engaged in a lively and wide-ranging debate over a great variety of issues impinging on education in geometry. This volume, consisting of position papers and recommendations, grew out of this meeting. Reactions to the contents of this volume are welcome and should be sent to me at the address below.

Growing out of the discussions at the meeting was a proposal to try to create a text which would serve as a model for the new directions for the college survey course. Again, the Sloan Foundation has generously provided funding for the writing and manuscript preparation for such a book.

I would like to thank all the mathematical participants in this project (including those unable to attend the conference, yet willing to endorse it recommendations) for their support, ideas, and enthusiasm. I would also like to thank the COMAP staff for its work in preparing for the conference, in running the conference, and in preparing this volume. Laurie Holbrook managed the manuscript preparation expertly and worked with Phil McGaw and his staff, who did their usual professional job. Special thanks go to COMAP's Executive Director, Solomon Garfunkel, for obtaining funding for this project, and for his many insights.

I look forward to a brighter future for geometry.

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Goals and Recommendations

In recent years, there has been a tremendous surge in research in geometry. This surge has been the consequence of the development of new methods, the refinement of old ones, and the stimulation of new ideas both from within mathematics and from other disciplines, including Computer Science. Yet during this period of growth, education in geometry has remained stagnant. Not only are few of the new ideas in geometry being taught, but also fewer students are studying geometry.

In March 1990, a group of college and university researchers and educators in geometry met to assess the directions of education and to make suggestions for invigorating it. These individuals represented a wide variety of branches of geometry as well as a wide spectrum of institutions. Discussions ensued on the causes of the decline in geometry education and on the steps that might be taken at all grade levels (K – graduate school) to energize the teaching of it. Special attention was given to the content of the survey course in geometry taught in many universities and colleges. This course has historically been taken by a large number of prospective high school teachers, and thus setting new directions for this course offers the hope of exposing future mathematics practitioners to new ideas in geometry, as well as for laying the basis for future changes in lower grades.

Despite the varied points of view expressed by the individuals who attended the conference, there was a broad core of common views, which, if implemented, can have a significant effect on geometry. This common core of views and recommendations is presented below.

Conference Recommendations

Future directions for the teaching of geometry (especially for implementation in the college/university survey course):

1. Geometric objects and concepts should be more studied from an experimental and inductive point of view rather than from an axiomatic point of view. (Results suggested by inductive approaches should be proved.)

2. Combinatorial, topological, analytical, and computational aspects of geometry should be given equal footing with metric ideas.

3. The broad applicability of geometry should be demonstrated: applications to business (linear programming and graph theory), to biology (knots and dynamical systems), to robotics (computational geometry and convexity), etc.

4. A wide variety of computer environments should be explored (Mathematica, LOGO, etc.) both as exploratory tools and for concept development.

5. Recent developments in geometry should be included. (Geometry did not die with either Euclid or Bolyai and Lobachevsky.)

6. The cross-fertilization of geometry with other parts of mathematics should be developed.

7. The rich history of geometry and its practitioners should be shown. (Many of the greatest mathematicians of all time: Archimedes, Newton, Euler, Gauss, Poincaré, Hilbert, Von Neumann, etc., have made significant contributions to geometry.)

8. Both the depth and breadth of geometry should be treated. (Example: Knot theory, a part of geometry rarely discussed in either high school or survey geometry courses, connects with ideas in analysis, topology, algebra, etc., and is finding applications in biology and physics.)

9. More use of diagrams and physical models as aids to conceptual development in geometry should be explored.

10. Group learning methods, writing assignments, and projects should become an integral part of the format in which geometry is taught.

11. More emphasis should be placed on central conceptual aspects of geometry, such as geometric transformations and their effects on point sets, distance concepts, surface concepts, etc.

12. Mathematics departments should encourage prospective teachers to be exposed to both the depth and breadth of geometry.

These recommendations are made in light of the belief that studying geometry can:

1. Illustrate how geometric mathematics is affecting modern life (e.g., compact disc recorders, CT scans, high resolution TV, image processing, robots, map projections, etc.).

2. Show the interplay of pure and applicable ideas (e.g., error correcting codes and sharp pictures of Uranus and Jupiter, knot theory to study DNA, etc.).

3. Provide a setting for principles of problem posing and problem solving.

4. Encourage visual thinking and reasoning (use of diagrams and models as modes of thought and problem solving).

5. Make clear how ideas developed for one application of mathematics are often transportable to other situations (e.g., getting a fire truck to a fire quickly and designing efficient paths for robots in a workspace).

6. Illustrate domains in which experiments can be done in mathematics and have students carry out such experiments (e.g., soap bubbles, tilings, mirrors to study symmetry).

7. Show how computers can be an aid to geometric thinking.

8. Illustrate how basic concepts such as distance, function, etc., are of use in a geometric setting.

9. Clarify the distinction between the mathematics of geometry and the geometry of physical space.

10. Make apparent the power of abstract thinking and the use of symbolism.

11. Show the rich history of geometry as a subject and the connection between geometry and other disciplines such as philosophy and physics.

12. Make clear how one part of mathematics makes contributions to other parts (e.g., the interplay between algebra and geometry, and combinatorics and geometry).

13. Illustrate what a mathematical proof means and to give examples of such proofs. (Note: there is no reason, however, to restrict the domain of such proof to theorems that appear in Euclid or similar results.)

14. Illustrate how ideas in mathematical modeling are of value in a geometrical setting and how geometric thinking is a tool for the mathematical model builder (e.g., use of graph theory to study problems in making deliveries to discrete locations, say oil to homeowner).

15. Foster better writing and communication skills when dealing with technical ideas.

Discussion

Although axiomatics account for a tiny portion of the research being done in geometry, the axiomatic point of view dominates the geometry being taught in high school and college. This has led to a curriculum where the geometry of triangles, circles, and quadrilaterals is developed from a very narrow point of view almost to the exclusion of other geometric ideas. Emphasis on axiomatic Euclidean (and in college, Non-Euclidean) geometry, deludes students into thinking that this must be the most important area for research and applications of geometry. Student "research" projects in courses of this kind lead them in directions that are out of step with current areas of interest in geometry and creates a gulf between active research geometers and educators and students in high school and college. The current geometric curriculum reinforces the view that geometry is merely a "technical" intellectual game without application or significant connection to the rest of the mathematical fabric. By breaking away from the current narrow curriculum, a variety of societally and mathematically desirable goals can be achieved.

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Geometry: Yesterday, Today, and Tomorrow

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Introduction

Despite the increased pace of exciting developments in both the theory and applications of geometry in the last 40 years, it appears that less geometry is being taught in college today than was taught in the recent or distant past. The purpose of this paper is to examine this "paradox" and to study how the teaching of geometry in college affects what geometry is and can be taught in high school, grade school, and graduate school mathematics.

Geometry in mathematics departments today

A perusal of recent college catalogues show mathematics departments listing (though not always regularly offering) a variety of geometric based courses: Graph Theory, Differential Geometry, Convex Sets and Geometric Inequalities, Combinatorial Geometry, Projective Geometry, Topology, etc. In addition to courses such as these, many mathematics departments offer a survey course in geometry under a variety of titles. These include College Geometry, Euclidean and Non-Euclidean Geometry, Topics in Geometry, Modern Geometry, Geometric Structures, etc. It will be convenient to refer to the first type of course as a Geometry Course and the second type as a Survey Course. (Geometry also enters the curriculum in a variety of other courses including Calculus, Linear Algebra, Combinatorics, etc.) Although it is rare to require either of these types of courses of students majoring in mathematics, it is not uncommon for many mathematics departments to require a Survey Course or some Geometry Course from those mathematics majors planning to teach mathematics in secondary schools. This type of requirement reflects the fact that the "traditional high school curriculum" includes a year of study of geometry in the 10th grade. Thus, the geometry taught in college is closely tied through teacher training to the geometry taught in pre-college mathematics. To explain the decline in the teaching of geometry in college requires a digression.

Why students major in mathematics in college

Most college mathematics majors fall into one of the following groups: students planning to enter graduate school to start in a program of doctoral studies in mathematics, students planning careers as high school (or sometimes intermediate school) teachers, students planning to pursue careers relating to computing, students planning actuarial careers, students planning to enter an "applied" masters degree program (this is usually a terminal degree that does not result in the student pursuing the doctorate degree), and "others." Especially at colleges in large metropolitan areas, high school mathematics teachers have traditionally constituted a significant portion of the total number of mathematics majors. With the downturn in mathematics majors that was seen in many colleges during the period 1972-1988, one saw a dramatic reduction in the number of students preparing for careers as secondary school teachers. This reduction is ostensibly attributable to several phenomena. First, the dramatic decrease in the number of students in the school system during the period meant that many teachers currently in the profession

were laid off. Second, the dramatic oversupply of mathematicians, engineers, etc. in the post-Apollo period made students wary of majoring in these subjects, and the high salaries paid in computer science siphoned away many students with interests in mathematics. Third, the salaries of high school mathematics teachers relative to other professions that potential teachers of mathematics could enter became eroded.

When the downturn in mathematics enrollments in general and mathematics secondary school teachers in particular hit our colleges, the effect on the teaching of geometry courses was especially extreme. This is clearly related to the fact that the major group of students taking survey courses, were future high school teachers. Even for geometry courses, loss of enrollment in high school teacher audiences resulted in decreased offerings. It is unfortunate that this diminished exposure to geometry for mathematics majors has come at a time of tremendous dynamism for geometry itself.

What is geometry?

Before continuing with more detailed discussions, it may be useful to explain how the term geometry has been and will be used in this essay. In attempting to clarify what is meant by the term geometry, it is clear that the word "geometry" means different things to different audiences, including subgroups of the mathematics community itself.

To lay people, geometry is the study of the space and the shapes that they see in the world around them. Most lay people's exposure to geometry is the simple material on classification of shape that they learn about in grade school and the exposure to "pseudo-axiomatic" geometry in high school. Much of high school geometry is still highly concerned with the axiomatics and the proving of Euclidean theorems in a manner that has come to be described as two-column proofs. This refers to a series of statements and the reason for the statements in a second column. In recent years, there has been a growing movement toward a more "inductive" approach to geometry, spurred on in part by the development of such software packages as the "Geometric Supposer." However, this movement has been nearly exclusively concerned with the metric properties of triangles, quadrilaterals, and circles. Thus, to the non-mathematician, geometry has a very narrow meaning. Obviously, "geometry" has much richer connotations to members of the mathematics community.

However, even within the mathematics community, geometry means a surprisingly diverse number of things to different people. To some, geometry refers to those portions of mathematics (and mathematical physics) that deal with the mathematical structure of space, thereby involving a large variety of deep mathematical tools such as operator theory, partial differential equations, and Lie groups. To others, it refers to differential geometry and the topology of manifolds. Yet other groups think of it as meaning (though not exclusively) the emerging body of ideas dealing with discrete geometrical structures. As diverse as the meaning of the word geometry is, a remarkably large portion of the subject can be introduced and profitably pursued with a minimum amount of background and formal study of mathematics. In this sense, geometry differs greatly from other parts of modern mathematics such as functional analysis, ring theory, logic, algebraic topology, etc.

Here the word geometry will be used in its very broadest sense of all aspects of mathematics where visual information, diagrams, models, and understanding of space are involved or put to use. For an attempt to catalogue the breadth of ground entailed by this viewpoint, see Malkevitch [1]. It is noteworthy that a variety of rapidly emerging areas within mathematics and computer science have a major geometric component. In order to see how geometry fits in the college curriculum of the future, it will be useful to examine the traditional relationship between geometry and other parts of mathematics.

Geometry's relation to mathematics

It is interesting to note that although many areas of mathematics have first been developed in geometric form, these areas have often matured when they were algebratized. Examples include synthetic Euclidean geometry, projective geometry, block designs, catastrophe theory, etc. As important as geometry is both to geometers and mathematics, as a separate discipline, it has never been in the mainstream of mathematics, once mathematics as a subject for study was institutionalized in universities and colleges. In a pre-World War II university or college, during the period when the roots of the current renaissance in geometry were being laid out at the research level, there were fewer Geometry and Survey Courses being taught than would have been the case from 1960-1975. Thus, a university during the 1920's or 1930's would have had courses in Analytic Geometry, Solid Analytic Geometry, Projective Geometry (perhaps in both synthetic and algebraic versions) and (old style) Differential Geometry. The wealth of geometry courses listed (though often untaught) at the college and university of today were uncommon then. In fact, at that time, no explicit survey course in geometry existed. (No equivalent of Howard Eve's pioneering Survey of Geometry (1963) with its curious forward- and backward-looking collection of topics existed before the War. The niche for high school teachers, trained then in "normal" schools or colleges, was filled by courses such as College Geometry or Modern Geometry. For a sample of the books of that era see Eves [1, p. 115]. Courses on convex sets, graph theory, groups and geometry, etc., virtually did not exist.)

Today, a standard introduction to mathematics for a graduate students pursuing a doctorate degree consists of a year of Real and Complex Analysis, a year of Abstract Algebra, and a year of Topology (with geometric aspects of the subject not necessarily emphasized). The teaching of topology often serves the role of hand-maiden for parts of Real and Complex Analysis. Judged by the dissertation titles that one sees listed in recent years by the American Mathematical Society, geometry is a relatively minor field at the fringes of most research. (Perhaps symptomatic of geometry's problems is that in the new 1990 mathematics subject classification list, the rapidly emergin area of computational geometry receives no listing.) The qualifier/preliminary examination system in place at most (especially as implemented at large) graduate schools discourages entry into "fringe" areas such as geometry. Thus, in a certain, very real sense, the study of geometry has not been in the mainstream of the training of professional mathematicians: those majoring in mathematics in college and going on to pursue doctoral studies in graduate school. Since there are not enough individuals who call themselves geometers to go around, most survey courses in geometry are taught by individuals with a narrow base of geometrical knowledge. Such individuals rely heavily on the geometry texts in print in teaching the Survey Course since teaching a course based on readings and their own knowledge base imposes a heavy preparation burden. (Geometry Courses are taught by the one member of the department who got the course listed in the catalogue in the first place, fall into disuse, or are taught by a "draftee.")

As noted before, many parts of mathematics have been developed in geometric form. Furthermore, a true renaissance of geometry has occurred in recent years. Examples of this ferment in geometric ideas include: the development of a new branch of mathematics, computational geometry; exciting breakthroughs in understanding the geometric structure of space (with resulting heavy cross fertilization with workers and ideas in mathematical physics); breakthroughs in the study of the mathematics involved in studying tiling problems for both the plane and higher dimensional spaces; an explosion of geometric ideas related to the theory of graphs with application to many areas of mathematics and operations research; dramatic new developments in the theory and application of the theory of knots; exciting connections between developments in the theory of dynamical systems and the geometry of sets (fractals); dramatic uses of geometrical methods in image recognition and processing; and use of geometric methods in the control and motion planning for robots and robot arms, to mention but a few of the more visible examples. This listing could easily be extended. Hence, it is increasingly unfortunate

that both teachers (already teaching and new ones being trained) and future researchers have not had available to them a vehicle for being exposed to the exciting new developments in geometry. Though geometric thinking itself may not be taught as part of the mathematical mainstream, geometry and geometric thinking is "infiltrating" mainstream mathematics more than ever before.

Geometry and Teacher Training

If American citizens are not to be raised as geometric illiterates, teachers in our grade schools and high schools will have to be broadly trained geometrically themselves. We have already examined the trend that new high school mathematics teachers entering our schools are few in number and have had less opportunity to be exposed to geometry than high schools teachers of earlier generations.

Many experiments are now being conducted to try to develop specialists to teach mathematics K-6. The need for "mathematics specialists" has been raised by the resistance of traditionally trained K-6 teachers to new developments and teaching methods in grade school. (Traditionally trained teachers in elementary school usually take a single course in mathematics as part of their teacher training. This course concentrates almost exclusively on the development of thinking about the base 10 number system, associated problems in addition, subtraction, multiplication, and division, and on measurement. This course rarely mentions any ideas in the area of geometry beyond simple taxonomy of simple shapes.) Emerging programs that urge specialists for elementary school to major in subject areas in college, as more reasonable preparation for teaching in grade school, will wind up subjecting such students to the very narrow type of geometry course now taught as a Survey Courses in our colleges. One of the few positive trends to note is that many teachers, both those planning to teach in high school or pre-high school environments, are being forced or encouraged to study the computer language called LOGO. Creative use of the LOGO language can permit students to be exposed to a wide range of open-ended, exploratory experiences with geometry.

Clearly, the Survey Course in geometry will play a large role in the exposure of future teachers to geometry. This is likely to become more so if future grade school mathematics specialists take this type of course. Thus it seems both wise and necessary for the mathematics community to significantly revamp the Survey Course. Such a change will be a service not only for future teachers and their students but for future researchers as well.

Goals in Changing the Survey Course as it Currently Exists

In attempting to change the content of the Survey Course, there are a variety of reasonable goals. Among these is the possibility of significantly changing the content of what is taught in high school by giving future high school teachers training in the geometry that might be part of a future high school geometry curriculum. Another goal is to encourage larger groups of students with interests in areas related to mathematics (e.g. computer science and engineering) to explore the many advantages that would accrue to them in being more broadly versed in geometric ideas. (The self-contained and quick starting nature of geometry makes this feasible.) A final goal might be to provide a rich variety of geometric concepts and tools for future research mathematicians both in traditional as well as emerging areas of mathematics, and to encourage more future research mathematicians to work in the area of geometry by exposing students to easily accessible unsolved problems.

Benefits of a Newly Constituted Survey Course

Although clearly geometry deserves to be studied for its own sake, many important objectives of mathematics study in general can show from studying geometry. Below is a partial list of some of the benefits of a revised geometry Survey Course (listed in random order):

1. To show how geometric mathematics is affecting modern life (i.e., compact disk recorders, CAT scans, HDTV [high definition TV], image processing, richer understanding of the geometry of space, robots, new types of maps, etc.).

2. To encourage visual thinking and reasoning (use of diagrams and models as modes of thought and problem solving).

3. To learn the interplay of pure and applicable ideas (e.g., error correcting codes and sharp pictures of Uranus and Jupiter, knot theory to study DNA, etc.).

4. To learn the distinction between the mathematics of geometry and the geometry of physical space.

5. To show the rich history of geometry as a subject and the connection between geometry and other disciplines outside of mathematics such as philosophy and physics.

6. To show how computers and specific software environments can be an aid to geometric thinking.

7. To foster better writing, verbal, and communication skills when dealing with technical ideas.

8. To illustrate how ideas in mathematical modeling are of value in a geometrical setting and how geometric thinking is a tool for the mathematical model builder (i.e., use of graph theory to study problems in making deliveries to discrete locations, say oil to homeowners).

9. To learn how ideas developed for one application of mathematics are often transportable to other situations (e.g., getting a fire truck to a fire quickly and designing efficient paths for robots in a workspace).

10. To obtain experiences in problem posing and problem solving.

11. To illustrate domains in which experiments can be done in mathematics and have students carry out such experiments (e.g., soap bubbles, tilings, mirrors to study symmetry).

12. To expose students to a variety of unsolved problems in geometry.

13. To learn how one part of mathematics makes contributions to other parts (e.g., the interplay between algebra and geometry, and combinatorics and geometry).

14. To illustrate how basic concepts such as distance, function, volume, etc., are of use in a geometric setting.

15. To illustrate the power of abstraction, special cases, and the use of symbolism.

16. To learn what a mathematical proof means and to give examples of such proofs. (Note: there is no reason, however, to restrict the domain of such proofs to theorems that appear in Euclid or similar results.)

Content for a New Survey Course

In attempting to design a new Survey Course in Geometry a variety of principles could be applied. Among these are that basic geometrical concepts and methodologies should be represented, that modern applications should be shown, that breadth as well as depth be respected, that a variety of geometric proof techniques be shown, and that a variety of different types of geometrical objects be examined. In addition to teaching a course based on significantly new content, I believe that the mathematics community should take advantage of new computer technologies (computer environments such as LOGO or Mathematica) and the use of videotape. For example, many applications of geometry are best introduced to a student in visual form using videotape rather than in written form. Appendix II shows various ideas for development of a video applications library to support existing and future text materials used in the teaching of geometry.

As a brief perusal of Malkevitch [1] quickly reveals, an exhaustive look at geometry in a semester sequence is not realistic. There is just too much attractive and important material. Any specific geometer is likely to have a somewhat different collection of topics and ordering for teaching these topics for a survey course from another geometer. However, I believe there is widespread agreement that the current course must be changed, moved in a direction away from axiomatics, and that any new course have a "core" of principles and content. In Appendix I, I have listed one of many possible approaches to both the content and organization of a new survey course that I have considered. Implementation of such a course will, I believe, be a major step toward attaining greater geometric literacy for teachers, the lay public, and mathematicians as well.

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Appendix I

Outline of some of the major topics to be covered in a geometry course of the future (listed in random order):

1. Combinatorial ideas vs. metric ideas
2. Convexity
3. Geometry of physical space (Relation to axiomatics)
4. Graph theory ideas
5. Computational geometry ideas
6. Symmetry, polyhedra, and tilings
7. Visual thinking
8. Area and volume (Bolyai-Gerwin, Hadwiger, Banach-Tarski)
9. Dynamical systems and fractals
10. Differential ideas
11. Applications
12. Role of dimension
13. Proof tools: Induction, infinite descent, examples, constructions, divide and conquer, etc.
14. Isomorphism concepts
15. Geometric transformations
16. Digital geometry
17. Packing and covering problems
18. Lattice point problems

Assumed prerequisites:

a. year of calculus or principles of mathematics course
b. knowledge of matrix notation and multiplication (but not necessarily of linear algebra)

Unit Outlines:

Unit I

A. What is geometry?
- i. Geometric pearls
  - a. Bolyai-Gerwin theorem
  - b. Euler's traversability theorem
  - c. Art gallery theorem (Fisk's proof)
  - d. Helly type theorems
  - e. Curves of constant breadth
  - f. Distance realization problems
  - g. Euler's polyhedral formula
  - h. Penrose tiles
  - i. Pick's theorem (lattice points)
  - j. Desargues' theorem

- ii. Visual thinking
  - Value of drawing diagrams
  - Value of constructing models
  - Geometry experiments (soap bubbles, etc.)
  - Computer environments

B. Different approaches to geometry
- i. Difference between metric geometry and combinatorial geometry
- ii. Axiomatic geometry and the geometry of physical space
- iii. Isomorphism concepts
- iv . Historical role of parallelism
  - a. Two space
  - b. Three space
  - c. Four space
  - d. Space-time
  - e. Surfaces embedded in three space
  - f. Dimension
- v. Deductive vs. inductive approaches to the study of geometry
- vi. Geometry and the computer
- vii. The relation of geometry to algebra and other parts of mathematics

C. Proof tools of the geometer
- i. Induction
  - On number of objects
  - On dimension
- ii. Infinite descent
- iii. Constructions
- iv. Algebra
- v. Arguments based on symmetry

Unit II

Types of Geometric Structures: (Graphs, Planes, Spaces, Block Designs, Convex Sets)

Graph Theory
- a. Traversability
- b. Trees
- c. Coloring problems
- d. Planarity
- e. Matchings
- f. Network Algorithms (shortest paths, flows, minimum-cost spanning trees, etc.)

Planes
- a. Affine planes
- b. Projective planes
- c. Hyperbolic planes
  (Infinite and Finite examples)
- d. Role of Desargues' "statement."

Space
- a. Euclidean, Projective, Hyperbolic space
- b. Axiomatics and Geometry of Space

Block Designs

Convex Sets
- a. Helly, Radon, and Cartheodory's theorems
- b. Minkowski addition
- c. Curves of constant breadth
- d. Geometric inequalities (isoperimetry)
- e. Lattice point problems
- f. Packing and covering problems

Unit III

Geometrical Transformations (viewed not as an approach to the theorems of Euclidean geometry, but for their own sake)

Transformations and their relationship to shape

Transformations and their relationship to metric properties (i.e., congruence)

Geometric transformations viewed geometrically

Geometric transformations viewed algebraically

Unit IV

Symmetry and Regularity
Polygons
- a. Plane polygons
- b. Convex polygons
- c. Self-intersecting polygons
- d. Packing and covering problems

Tilings
- a. Tilings with regular polygons
- b. Tilings with convex polygons
- c. Symmetry properties of tilings
- d. Aperiodic tilings
- e. Penrose tilings

Polyhedra
- a. Regular polyhedra
- b. Archimedean polyhedra
- c. Combinatorial properties of polyhedra
  - i. Euler's formula
  - ii. Steinitz's Theorem
- d. Minkowski addition aspects of polyhedra
- e. Graphs of polyhedra
- f. Tilings in space

Symmetry Groups
- a. Symmetry groups of tilings, patterns, fabrics, etc.

Unit V

Area and Volume
Equidecomposability
Role of Archimedes' Axiom
Squaring the circle
Banach-Tarski Paradox
Dynamical systems and Fractals

Unit VI

Computational geometry
Triangulations
Voronoi Diagram
Sweep line methods
Convex Hull
Principles of design for geometric algorithms

Unit VII Topological ideas

Geometry of Surfaces
- a. Orientability (Moebius band)
- b. Torus
- c. Klein bottle

Knots
- a. Geometric transformations of knots
- b. Classification of knots

Unit VIII Geometric Optimization Problems

Linear Programming
Isoperimetry
Packing and Coverings
Network Optimization

Unit IX History of Geometry

Geometry in the ancient world
Geometry during the Renaissance
Geometry up to the 20th century
Geometry in the 20th century
Note: There should be biographical material about the great contributors to geometry, including, where possible, portraits or photographs.

Unit X Applications of Geometry

Probably applications should be sprinkled in and included in an integral manner with the other parts of the materials being developed. However, here are some particularly topical areas that might be mentioned (see Appendix II for additional examples):

Robotics
Computer vision
Computer graphics
Solid modelling
Operations research

Note 1: Unsolved problems in geometry would be mentioned throughout the course.

Note 2: For bibliographic references in support of a wide variety of classical and recent topics, see Malkevitch [1].

Appendix II

Examples of situations to be developed on videotape:

EDGE TRAVERSAL

Situations:

- Curb inspecting
- Street sweeping
- Garbage collection
- Mail delivery
- Advertising circular delivery
- Painting line down center of roads
- Snow removal
- Parking meter collection and enforcement
- Police or museum guard patrol routes
- Pipe, wiring, or duct inspection

Mathematics:

- Graphs as models
- Euler's traversability theorem
- Chinese Postman Problem
- Johnson and Edmond's algorithm
- Deadheading and repeated edges

Practitioners:

- U.S. Postal Service
- Sanitation Department
- Department of Parking Enforcement
- University Operations Research Departments (MIT, Maryland, Stony Brook)
- ATT Bell Laboratories; Bell Communication Research

VERTEX TRAVERSAL

Situations:

- Meals on wheels
- Deliveries to supermarkets, restaurants, etc.
- Garbage pickup from industrial sites
- Machine inserter schedules
- Computer solution of jigsaw puzzles
- School bus routes
- Camp pickup routes
- Parcel post delivery and pickup
- Pizza delivery
- Special delivery of mail
- Pickup of coins from pay telephone booths

Mathematics:

- Graphs as models
- Hamiltonian circuits in graphs
- Traveling salesman problem
- Asymmetry of costs
- Complexity
- K-opt methods
- Greedy algorithms
- Vehicle routing problems
- Clarke-Wright algorithm

Practitioners:

- Sanitation department
- U.S. Postal Service
- Federal Express
- Parcel Post
- School Boards
- Camps
- University Operations Research Departments (MIT, Stony Brook, Maryland)
- ATT Bell Laboratories; Bell Communications Research

VORONOI DIAGRAMS

Situations:

- District planning
- Drainage regions
- Market structure (anthropology)
- Robot motion planning

Mathematics:

- Computational geometry
- Perpendicular bisector
- Convex set
- Convex hull
- Concurrence, concyclic points
- Line sweep algorithms

Practitioners:

- University Mathematics and Computer Science Departments (Smith College, Courant Institute, U. Illinois, Princeton, Rutgers)

ROBOTS (MOTION PLANNING)

Situations:

- Industries which employ mobile robots
- Planetary surface exploration

Mathematics:
- Graphs as models
- Visibility graphs
- Shortest path algorithms
- Minkowski addition
- Parallel domains
- Vision

Practitioners:
- General Motors, Ford, Chrysler, etc.
- Universities: (MIT, Yale, Courant Institute (NYU), Stanford)
- ATT Bell Laboratories

Note: other aspects of robotics also involve geometrical ideas. These include the local motion planning of the gripper of a stationary robot.

BIN PACKING

Situations:
- Machine scheduling (independent tasks)
- Organizing computer files on disks
- Advertising breaks
- Want advertisements in newspapers

Mathematics:
- Packing problems
- Heuristic algorithms
- Measures of efficiency
- Time space tradeoffs
- Complexity
- Simulation

Practitioners:
- Operations Research Departments (Berkeley)
- ATT Bell Laboratories

DISTANCES

Situations:
- Car travel
- Urban distance
- Biology (Evolutionary trees)

Mathematics:
- Taxicab metric
- Abstract properties of distance
- Sequence Comparison
- Levenshtein Distance

Practitioners:
- ATT Bell Laboratories

SHORTEST AND LONGEST PATHS

Situations:
- Firetruck and ambulance routing
- Building construction
- Space program (flight planning)
- Robot motion planning

Mathematics:
- Graphs, digraphs, and weighted graphs and digraphs
- Dijkstra's algorithm
- Critical Path Method

Practitioners:
- Operations researchers

MINIMUM-COST SPANNING TREES

Situations:
- Synthesis of communication networks
- Road planning

Mathematics:
- Graphs as models
- Trees
- Spanning trees
- Kruskal's algorithm
- Prim's algorithm
- Greedy algorithm

ERROR CORRECTING CODES

Situations:
- Compact disk players
- Computer codes
- Space programs
- HDTV

Mathematics:
- Binary sequences
- Distance (Hamming distance)
- Matrices
- Information content

Practitioners:
- Compact disk manufacturers (Philips)
- Universities: California Institute of Technology, MIT
- ATT Bell Laboratories

COLORING PROBLEMS
Situation:
- Scheduling committees, final examinations, railroads
- Fish tanks and animal confinement patterns
- Maps
- Placement of guard in art galleries

Mathematics:
- Graphs as models
- Vertex colorings
- Face colorings
- Edge coloring
- Complexity

Practitioners:
- Universities with graph theory specialists
- ATT Bell Laboratories; Bell Communications Research

DATA COMPRESSION
Situations:
- Image transmission and storage
- Text transmission and storage

Mathematics:
- Binary numbers
- Digitalization of text and images
- Huffman codes
- Fractal methods

Practitioners:
- Universities (MIT, Georgia Institute of Technology)
- ATT Bell Laboratories; Bell Communications Research
- NASA

GEOMETRIC TRANSFORMATIONS AND SYMMETRY
Situations:
- Computer graphics
- Analysis of fabrics
- Analysis of archeological facts
- Cartography
- Analysis of art (Escher paintings)

Mathematics:
- Group theory
- Functions and transformations
- Strip groups
- Wall paper groups
- Color symmetry

Practitioners:
- University mathematicians

UNFOLDING POLYHEDRAL SURFACES
Situations:
- Catching a spider on the wall of a cube
- Drawing a map of a spherical surface
- Unfolding the surface of the brain
- Layouts for packages

Mathematics:
- Development of polytopes
- Projection mappings
- Distance on polyhedral surfaces

BLOCK DESIGNS
Situations:
- Drug testing
- Agricultural productivity
- Scheduling workers
- Scheduling tournaments

Mathematics:
- Finite geometries
- BIBD's
- Orthogonal Latin Squares

Practitioners:
- Universities: Ohio State University, CAL Tech
- ATT Bell Laboratories

MATHEMATICAL PROGRAMMING
Situations:
- Blending gasolines
- Blending juices
- Manufacture of processed foods
- Scheduling
- Shipment of goods
- Vehicle routing
- Hospital management
- Portfolio management

Mathematics:

- Linear programming
- Integer programming
- Linear inequalities
- Solution of linear equations
- Network flows
- Transportation problem

Practitioners:

- Universities: Rutgers, Princeton, Stony Brook
- ATT Bell Laboratories
- Oil companies, airlines, car companies, defense industries

ART GALLERY THEOREMS
Situations:

Surveillance in museums, banks, and military installations

Mathematics:

- Convex sets
- Types of polygons
- Triangulations
- Colorings

Practitioners:

- Computational geometers

EUCLIDEAN GEOMETRY
Situations:

- Length of carpet remnants
- Time remaining on a partially used tape

Mathematics:

- Geometry of the circle (circumference)
- Areas and perimeter concepts
- Isoperimetry

The situations above are organized by mathematical theme. Other approaches also exist, in particular, showing applications of geometry to a particular subject area. Several examples of this are given below:

Applications of Geometry to Business:

1. Traversability problems
2. Minimum-cost spanning trees
3. Facility location problems
4. Coloring problems (scheduling problems)

Applications of Geometry to Medicine:

1. CAT scanners and other medical imaging systems
2. Kidney stone machines
3. Brain mapping studies

Applications of Geometry to Biology:

1. Structure of the gene (intersection graphs, interval graphs)
2. Food chains, niche spaces, competition (intersection graphs)
3. Ecology (fractals)
4. Shape of biological forms (isoperimetry)

Applications of Geometry to Chemistry:

1. Quasicrystals (Penrose tiles, crystallography)
2. Dynamics of chemical reactions (dynamical systems)

Applications of Geometry in Communications

1. Synthesis of communications networks
2. Vulnerability of communication networks
3. Phone exchange systems
4. Error correction methods (Error correcting codes)
5. Data compression (Compression of text and images)
6. Digitalization of images
7. Image processing (filtering etc.)

Applications of Geometry in Social Science

1. Analysis of fabrics, designs and pottery (anthropology) - groups, symmetry patterns
2. Kinship systems (anthropology) - graph theory
3. Mobility (sociology) - Markov chain digraphs
4. Equilibrium analysis (economics) - dynamical systems

Visualization and Visual Thinking

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Introduction

The debate of not so long ago over whether geometry is dead seems quaint today, when so many disciplines, including our own, are being revolutionized by computer graphics. Even research in the highly abstract fields of mathematics into which geometry was said to have been absorbed is radically changing direction because of this powerful tool. Many mathematicians say that computer graphics are now indispensable for their research. We no longer need to worry about geometry's health; it is certifiably alive and well.

I was tempted to say that our task at this conference is to find a way to restore geometry to its rightful place in the college curriculum, but what is its rightful place? In mathematics education, as well as in our research, geometry can now play roles it never played before.

In the following remarks, I will attempt to formulate my opinions about visualization and visual thinking. They are necessarily tentative: to speak more authoritatively on this very complex subject one ought to have some expertise in pattern recognition theory, in the physiology of perception, or at least in computer graphics, and I have no experience in any of these fields. But I have taught college students for many years, and my research has been in the very visual field of mathematical crystallography; it is this experience that informs my views. These views are not original; in fact, this paper is essentially a string of quotations from other sources. The important point is that together these sources show that the case for making visual thinking a central tool of geometry is not vague, but is supported by a strong body of research.

Visual Thinking vs. Visualization

Let us begin by making a distinction between "visual thinking" and "visualization." In popular speech, the latter means "space perception"; indeed, I have never read or heard a discussion of visualization in geometry that did not use these expressions interchangeably. For example, I often hear teachers lament their student's apparent inability to "visualize" a cube when presented with a planar drawing of one.

On the other hand, "visual thinking" is a much broader term. We are thinking visually when we sight-read a sheet of music, as well as when we decide by inspection that a graph is bipartite or that a diagram is a representation of a cube. In other words, visual thinking is what we are doing when we rapidly recognize and "automatically" manipulate symbols of any kind. Space perception, on the other hand, is the mental reconstruction of representations of three-dimensional objects.

I will not discuss visualization in this paper, except for the following brief remarks.

Most discussions of space perception focus on who can do it. I suggest we not let ourselves get trapped in that thicket. Evidently, space perception does vary somewhat from individual to individual, but the degree to which this may be so should not be our concern here. There are many reasons to steer clear of that old debate. Historically, it has inspired a great deal of nonsense, such as Felix Klein's suggestion at the turn of the twentieth century that space perception might be a characteristic of the "Teutonic race," and the current dispute over whether boys have this quality to a greater degree than girls. In any case, it is likely that very soon the question will not be debated any more: the computer will have made it obsolete.

All that needs to be said on the subject of "who can visualize?" is contained in the following quotation from *Visualization: The Second Computer Revolution* by Friedhoff and Benzon (but replace "visual thinking" by "visualization"):

> "Two findings are quite persistent in research on visual thinking. The first is that the way in which people think visually is different from individual to individual. The second important finding is that the degree to which individuals rely on visual thinking is, like almost every other measurable characteristic, distributed in the population. Some individuals think more visually than others.
>
> "An important aspect of visualization with the computer is that it tends to even out these differences among people. ... visualization need no longer be a solitary inner experience. The computer makes it possible for groups of individuals, even if they are separated by great distance, to collaborate in visual exploration, whether in the artistic, design, or scientific spheres. The computer democratizes visual thinking."

Finally, I do not believe that visualization per se is an appropriate subject for instruction at the college level. If we structure our courses so that students wrestle with real geometrical problems, this will almost always engage them in drawing diagrams, making models, or using computer graphics. Then we will not need to worry about their visualization skills.

Visual thinking is real thinking

To study the role of images in geometry education, we should begin with the larger question of visual thinking. We all think visually to such an extent that we take it for granted, but we should not. Properly exploited, visual thinking can revolutionize the way we do and teach geometry. The first step is to realize not only that we should take it seriously, but also that we do take it seriously.

An example from another discipline may be helpful in understanding the way in which we tacitly acknowledge that visual thinking is real thinking. A colleague of mine in the history department told me the following story. A student who was very bright had surprisingly great difficulty with everything relating to geography: she simply could not learn to read maps. The faculty was quite puzzled, because the girl excelled in everything else. My colleague eventually learned that the student was color blind! Because of this handicap, which evidently caused relatively little difficulty in other areas of her life, she could not grasp spatial relations among regions in the instantaneous way the other students could. The information encoded geometrically in colored maps could not be translated into nonvisual language, and thus maps were almost incomprehensible to her.

Obviously, visual thinking goes far beyond map-reading. I have already cited reading music as an example. Even in mathematics we use it in areas when geometry plays no role at all. For example, in his famous lecture "Mathematical Problems," David Hilbert noted that,

> "Arithmetical symbols are written diagrams and geometrical figures are graphic formulas.... The agreement between geometrical and arithmetical thought is shown also in that we do not habitually follow the chain of reasoning back to the axioms in arithmetical, any more than in geometrical discussions. On the contrary, we apply, especially in first attacking a problem, a rapid, unconscious, not absolutely sure combination, trusting to a certain arithmetical feeling for the behavior of the arithmetical symbols, which we could dispense with as little in arithmetic as with the geometrical imagination in geometry."

In other words, even in ordinary arithmetic, our thought processes utilize symbols directly, without explicitly translating them into words or concepts which the symbols represent. And come to think of it, isn't that exactly what we do when we read?

In his well-known book on psychology of art, *Visual Thinking*, Rudolf Arnheim states that,

> "The cognitive operations called thinking are not the privilege of mental processes above and beyond perception but the essential ingredients of perception itself. I am referring to such operations as active exploration, selection, grasping of essentials, simplification, abstraction, analysis and synthesis, completion, correction, comparison, problem solving, as well as combining, separating, putting in context.... Visual perception is visual thinking."

Arnheim is describing, in fact, much of what we call mathematical thinking. (For an illuminating discussion of this subject, see *The Emperor's New Mind*, in which Penrose explores nonverbal aspects of mathematical thought.)

Yet there is a suspicion "out there" that visual thinking is not rigorous. Arthur Loeb describes an experiment in nonverbal computer learning that he conducted some years ago with a colleague:

> "In carrying out these experiments, we took a group at random – for example, some people who had a summer job in an office there – and we gave them a task. We always made certain that we gave them a test before they went through the program, and afterwards we gave them the same test again and compared test scores on the pre-test and on the post-test. We found something very interesting. The task at that time was to take the upper case letters of the alphabet, all of them – A, B, through Z – and to put them into different classes according to whatever symmetry they had. For instance, the A and T both have vertical mirrors, so we wanted them put in the same box. The letter N and the letter S both have only two-fold rotational symmetry so they would together go into a second box. It was remarkable that in the pre-test, of course, they had no idea how to classify these letters, and so they scored very low. After an hour, they did extremely well – most of them infallibly classified these letters. But when we asked what they had learned, they replied, 'Why, nothing.'.... So we remarked, 'But you must have learned something. You were able to do that task afterwards, and not before.' The reply was, 'Oh, we did it intuitively.' Now you see, they hadn't learned anything; they did it intuitively, but they didn't have that intuition before! At 12 o'clock they didn't have the intuition, but at one o'clock they did. It gave me considerable insight into what people call intuitive. I think that a lot of what is called intuitive is, in fact, nonverbalized knowledge.... I think that is very significant in this whole notion of visual thinking."

Arnheim remarks that,

> "The arts are neglected because they are based on perception, and perception is disdained because it is not assumed to involve thought. ... The neglect of the arts is only the most tangible symptom of the widespread unemployment of the senses in every field of academic study. What is most needed is not mere aesthetics or more esoteric manuals of art education, but a convincing case made for visual thinking quite in general."

Many people in the visual arts agree, and Loeb and others in the field of "design science" have developed very interesting courses of study whose goal is to develop "a nonlinear language, a language that does not consist of linear strings of symbols."

These same concerns are being voiced today in the mathematical community. For example, in a recent issue of the *Notices*, Jon Barwise asks,

> "Why shouldn't graphical representations share the role in mathematical proofs traditionally reserved for linguistic representations (i.e., sentences)? More specifically, why shouldn't diagrams and other forms of graphical representations be used as essential constituents in the statement of a theory, or in its proof, or both?
>
> "This is a logical heresy, of course, as we are all taught at our mother's knee when we study geometry. But why should the linguistic representations have such an exclusive hold over other ways of representing mathematical objects and mathematical information? .... Upon examination, one finds that all the hazards associated with visual representation are also present in linguistic representations. And the computer is beginning to provide us with tools to overcome the problems that have given diagrams and other visual representations such a bad name in mathematics."

Barwise's questions are very much to the point. The answers are, I suspect, deeply rooted in the culture of our time. Is it only an accident that the development of abstract mathematics paralleled the rise of atonal music and abstract art? We should seek the answer to this question in Thomas Mann's *Dr. Faustus*, as well as in the writing of Hilbert and Bourbaki. In this century, the ability to conduct mathematical research without recourse to illustrations, or at least to pretend to do so, became the very hallmark of acceptability into the ranks of the blessed. The sources of this phenomenon most likely lie in some complicated knot of sociology, psychology, and cultural imperative. This would make an excellent topic for a doctoral thesis in the history of mathematics.

To be fair, it should be acknowledged that part of the widespread distrust of pictures and arguments based on them was not always unreasonable: there probably was good reason to be concerned about the subjective element in visual arguments. And certainly we have all seen enough badly drawn pictures to be reluctant to teach concepts that rely on them. But such worries predate the development of first-rate computer graphics. Moreover, there is a large body of interdisciplinary scientific research that gives strong support to the view that visual representations can indeed be placed on an equal footing with linguistic ones.

Some of this research is being carried out not by mathematicians or mathematics educators, but by scholars concerned with the linguistic abilities of the deaf. It is described in Oliver Sacks' new book, *Seeing Voices: A Journey Into the World of the Deaf*, and it is highly relevant to our topic, although mathematics per se is never discussed. (Sacks does mention, as an aside and in another context, that at Gallaudet University, the world's only liberal arts college for the deaf, the only department with a

majority of deaf faculty is mathematics. Is this coincidental?)

Seeing Voices

Seeing Voices is, among other things, a book about visual thinking. A large portion of it is devoted to a discussion of Sign, which Sacks asserts is a natural language of the deaf, distinct from Signed English, or Signed French, say, in which the words of a spoken language are communicated through signs. Sign is a space-time language communicated by rapid sequences of complex motions in three-dimensional space.

Sacks describes a visit to a community in Martha's Vineyard in which there was hereditary deafness, and where nearly everyone, deaf or hearing, was fluent in Sign. What he saw there was a revelation to him:

> "And, speaking to one of the very oldest there, I found one other thing, of great interest. This old lady, in her nineties, but sharp as a pin, would sometimes fall into a peaceful reverie. As she did so, she might have seemed to be knitting, her hands in constant complex motion. But her daughter, also a signer, told me she was not knitting, but thinking to herself, thinking in Sign. And even in sleep, I was further informed, the old lady might sketch fragmentary signs on the counterpane – she was dreaming in Sign.... Sign, I was now convinced, was a fundamental language of the brain."

Most of the reviews of Sacks' book have focused on the world of the deaf that he portrays, and on the argument about whether it is better to teach deaf children to speak with their lips or with their hands. But for our purposes the most important part of the book is the extensive technical discussion that is the rationale for the last sentence in the quotation above. Sacks notes:

> "Notions that 'the sign language' of the deaf is no more than a sort of pantomime, or pictorial language, were almost universally held even thirty years ago.... It was Stokoe's genius to see, and prove, that it was nothing of the sort; that it satisfied every linguistic criterion of a genuine language, in its lexicon and syntax, its capacity to generate an infinite number of propositions.... Stokoe was convinced that signs were not pictures, but complex abstract symbols with a complex inner structure."

Later in the book, Sacks points out that,

> "The single most remarkable feature of Sign – that which distinguishes it from all other languages and mental activities – is its unique linguistic use of space. The complexity of this linguistic space is quite overwhelming for the 'normal' eye, which cannot see, let alone understand, the sheer intricacy of its spatial patterns... What looks so simple is extraordinarily complex and consists of innumerable spatial patterns nested, three-dimensionally, in each other."

Sacks cites evidence that the view that our right and left brains deal with different cognitive tasks is an oversimplification. It is more likely that,

> "The right hemisphere's role ... is critical for dealing with novel situations, for which there does not yet exist any established descriptive system or code – and it is also seen as playing a part in assembling such codes. Once such a code has been assembled, or emerged, there is a transfer of function from right to left hemisphere, for the latter controls all processes that are organized in terms of such grammars or codes."

Like spoken languages and other descriptive systems such as music and mathematics, Sign is at first a "right-brain" activity that subsequently becomes routine as a left-hemisphere function.

Like any spoken language, Sign is most easily acquired in the first few years of life. In this respect, children born deaf have a distinct advantage over children who become deaf after they have learned to speak. Still, like any other language, it can be taught and learned.

Toward a Dynamic Geometry

Sacks does not discuss the implications of his study for mathematics education, but it suggests to me that the possibilities for giving models and images a central role may be far greater than even the most enthusiastic among us have supposed.

In the first place, it suggests that we have not begun to tap the geometrical imagination and abilities of our students or even ourselves. We can do much, much more, by using three-dimensional models of many kinds and sophisticated computer graphics to create a visual, spatial language in which we can be as fluent as we are in speaking, reading, and writing. Second, there is no reason whatsoever why graphical representations should not share the role in mathematical proofs traditionally reserved for sentences. Indeed, with graphical representations, we can go far beyond this traditional role. Models and three-dimensional images on the computer screen need not be static; like Sign, they can utilize all the possibilities inherent in both space and time. Of course, the dynamic capabilities of the computer are already being exploited; what has not yet been exploited is its potential role in education.

As we rethink the role that models and computer graphics can play in geometry education at all levels, we must keep in mind that it is not enough just to say that we need more of them. It is important that models and graphics not become the mathematical equivalent of Signed English. That is, we should not use them only to illustrate the things we already teach, to "back up" abstract ideas. Instead, we should think about broadening geometry courses by a fuller utilization of visual thinking.

I will cite just one among several intriguing contemporary examples. Most of us may not think of calculus as a geometrical subject, but it was highly geometrical before the nineteenth century emphasis on rigor came to dominate the way we think about it, and the way we teach it. Today many mathematicians are increasingly dissatisfied with their calculus teaching, and believe that the standard course is in need of fundamental change. The National Science Foundation evidently agrees, since it has awarded grants to many different teams to restructure it. I think I am correct in saying that all of the reformers are trying to integrate the computer into the course work to a greater or lesser extent, and that in most cases computer programs are being written to illustrate the processes that we have always taught with paper, pencil, and chalk (the idea of the derivative as a limit, the Riemann integral, and so forth): that is Signed Calculus.

But one of these teams (some of my colleagues in the Five Colleges) is doing something quite different. Their project, *Calculus in Context*, engages students in the experimental solution of real-life, mathematically messy problems from the very first day, using a combination of interactive computer graphics and numerical methods. The students learn to understand "what is going on" graphically; what they see is what is going on. Visualization is the subtext of the entire course: analysis explains the graphics, not vice versa. For example, I recently visited a Calculus II class which was studying the Fourier transform. I walked in a skeptic; I walked out impressed with what I had seen. I suspect that the calculus students, with a clear visual understanding of what the transform is and why it works, understood the concept better than many other, more advanced students taught through traditional methods ever do.

It will take a lot of work on our part to convince geometry teachers (K – Ph.D.) that visual thinking is not a crutch to be discarded as early as possible. We must refute the deeply ingrained belief that the purpose of models and illustrations is to help the student along the road to abstraction. Whenever I talk with teachers, I find ready agreement that students should have access to "manipulatives" and that computer graphics are great fun. But when I probe a little further, I inevitably hear that these things are especially important in the early grades, and for slow learners; bright older students don't need them because they can do "real" mathematics.

But in fact visual thinking is a crucial component in the psychology of mathematics, not an adjunct to it. This becomes obvious when we remember that much of the history of mathematics is the history of mathematical notation; finding compact, versatile mnemonic symbols has been to the mathematician what instrument technology has been to the experimental scientist. (Try calculating with Roman numerals, or integrating without an integral sign; recall the debate over Leibniz's vs. Newton's notation for the derivative; imagine doing contemporary mathematics – or physics or economics – without the matrix.)

Today, for the first time, we have the capacity to bring geometrical symbols into the realm of accepted mathematical discourse. Computer technology enables us to extend the symbols with which we think beyond those which can be set linearly in type to include geometrical forms, even forms in motion. The geometry course we are planning can lead the way.

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Robotics in Geometry Courses

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Principles for Designing Geometry Courses

Over the last two decades (at least), there has been no useful consensus on what a geometry course should consist of. I myself have taught the same course in geometry a number of different ways: convex sets, polyhedra, axiomatic Euclidean geometry, axiomatic non-Euclidean geometry, etc. No other course in the mathematics curriculum has such an identity crisis. I do not believe that geometry courses will survive the competition for space in the curriculum unless a common and attractive agenda for the course can be set forth loudly and clearly.

If it doesn't matter what we teach, then it doesn't matter if we teach it at all.

But we cannot create a consensus as a political act. We need a set of objectives and principles for a geometry course which imply a curriculum. A number of the criteria which seem to be important to geometers today unfortunately do no imply any particular curriculum. These criteria include notions such as:

1. Students need practice in visualization.
2. Geometric interpretations can be helpful in other courses.
3. Geometry has historical interest.
4. There's lots of aesthetically pleasing material in geometry.

(All these things are true of course – but that's not the point!)

One principle which does imply curriculum and is in common use is to try to prepare future high school math teachers for teaching geometry. There is nothing wrong with this objective, but it should not become overriding. At the college level, we should be the leaders and not the followers.

However, in thinking about the high school curriculum, one is reminded that this curriculum has aims other than to prepare students for majoring in mathematics. A wide variety of occupations require some understanding of geometry, including physics, chemistry, many engineering professions, architecture, urban planning, and so on. Indeed, we may take a hint from this fact. It would be possible to provide a geometry course which has much material of interest to students not majoring in mathematics. This might help counter the deplorable trend in which upper division mathematics courses are offered by other departments because they feel badly served by mathematics department offerings.

Two principles are in common use for determining curricula for other mathematics courses:

1. Teach things which have important applications.
2. Teach things which are on a direct path to modern research.

We could use these principles also. I would suggest a third which is peculiar to geometry (although it could be regarded as a subcategory of number 1):

3. Since we live in a geometric space amidst other geometric objects, we should teach about that space and those objects.

One category of subjects that matches up well with these criteria is the computational geometric questions stimulated by robotics. Robotics is an appropriate application for a geometry course: less transitory than most applications, and destined to have a major impact on the way we live. In the coming decades, we may see robot vacuum cleaners, lawn mowers and other household machines; robot cars which enable us to read, sleep or do mathematics on the way to work; a variety of assistants for the handicapped or infirm; finally, robots in factories in increasing numbers.

Robotics applications do not fall into a single category of geometry; they exist in the fields of graph theory, convex sets, geometric transformations, algebraic geometry, differential geometry, etc. Consequently, robotics applications could be sprinkled throughout a geometry book. Alternatively, a separate chapter could be provided to try to tell a more coherent story. The latter strategy would be facilitated if some of the basic geometric ideas had been established in earlier chapters.

It is essential to understand that almost everything in robotics usually comes down to the need to calculate something – and preferably calculate it quickly. Thus, coordinate methods and the algorithmic point of view are essential. As a result, the subject of robotics merges imperceptibly into linear algebra, analysis, and the complexity of algorithms. Since we do not propose to write a robotics book, we can't follow the subject far into all these areas. But the ties to these other subjects should be seen as attractive rather than unfortunate.

In the following sections, we discuss four geometric aspects of robotics which could be included in an elementary geometry book: motion planning for mobile robots, computer vision, linkages and their configuration spaces, and computation of robot arm kinematics. The geometric subjects that come up in these areas include:

1. convex sets and polyhedra
2. shortest path algorithms
3. rigid motions
4. orthogonal and perspective transformations
5. homogeneous coordinates
6. computational complexity of algorithms

Motion Planning for a Mobile Robot

Robot motions occur in continuous space, but in order to compute them it is often necessary to discretize space somehow. Consider a "point robot" moving on the floor (the plane) from point s to point g amid convex polygons as in **Figure 1**. One discretization is the visibility graph, obtained by connecting mutually visible points in the union of $\{s, g\}$ and the vertex set. This then brings us to the problem of finding a shortest path in a graph from one point to another.

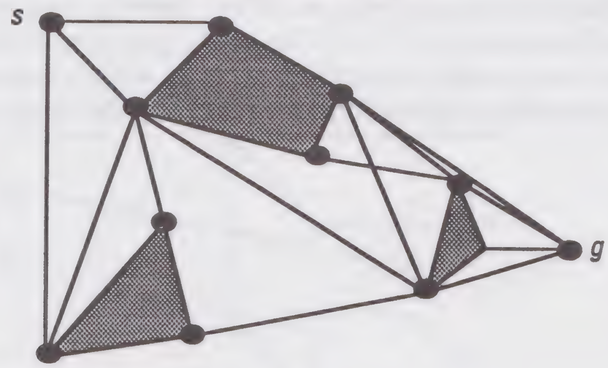


Figure 1.
The visibility graph for traveling from *s* to *g*.

Moving along the visibility graph is dangerous because one touches the obstacles at corners. A graph that somehow "keeps its distance" from the obstacles would be desirable. Students could be asked for ideas for methods of generating such a graph.

Constructing the visibility graph in the minimum number of steps also provides interesting questions. If there are n points (including s and g) it would appear that all n^2 line segments determined by pairs of points must be tested for intersection against all polygon edges. But this can be reduced substantially by taking advantage of geometric structure. [Shamos and Preparata]

Computer Vision

Another category of discretizations involves images captured by the robot's vision system. If we assume, for simplicity, a black and white world, such an image is nothing more than a grid of squares (usually called pixels), some of which are black and some white (**Figure 2**). Imagine, for example, dark objects moving down a light colored assembly line and suppose the camera needs to determine how many objects are in the field of view at some instant in time. This can be reduced to finding the "connected component" in which a given dark pixel lies. (One then counts the components.) The input to the connected component algorithm is a set of coordinates for the pixels which are dark. How could such an algorithm work? That's not such a hard question. More interesting is the prior question of when we should regard two pixels as connected. For example, in Figure 2, it seems clear that the "L" shaped objects is connected, but what about the four pixels making up the "Y" shaped object? Depending on how one answers this, one gets different answers from a connected component algorithm.

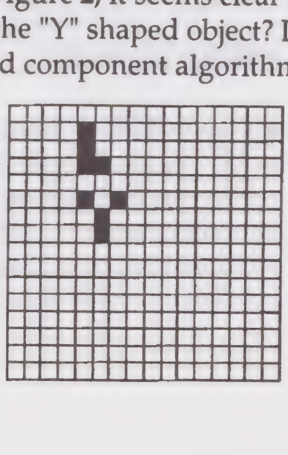


Figure 2.
An image of pixels gathered by a computer vision system.

Now let's back up a step and ask how an image is formed in the first place. The formation of images, in a camera or the retina of the eye, can be roughly described as a perspective transformation – a classical friend from projective geometry. **Figure 3** shows the perspective transformation created by a pinhole camera. Such a camera, although extremely simple, captures the essential features of image formation in real cameras for our purposes. Each point P in the "real world" outside the camera which has an unobstructed line of sight to the pinhole (lens) creates an image of itself, P', on the image surface. The image surface, in this pinhole camera, is the back plane of the box and represents the film, retina, or charge coupled device in the imaging mechanism being modeled. Since the image surface is a plane, what we have is a perspective transformation, $\pi: R^3 \Longrightarrow R^2$. Since the geometry of the camera (location of pinhole, size of box, etc.) is known, the equations describing π can be written down explicitly as soon as one chooses a coordinate system in the three-dimensional world and in the image plane.

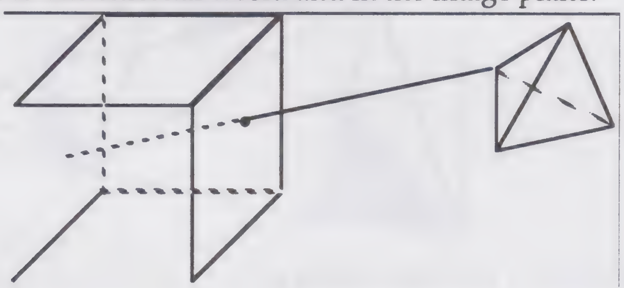


Figure 3.
A pinhole camera imaging a tetrahedron.

Given an object, B, in the three-dimensional world R^3, which features of B are distorted and which are preserved? This is the classical question about invariants of a projective transformation, but it is also useful background information for writing computer vision software. For example, suppose one wants to recognize table tops in images. Table tops are rectangles, but we cannot depend on the image of a table top being a rectangle. However, lines are preserved, so it is reasonable to look for quadrilaterals in order to find table tops.

A fundamental problem of computer vision is to determine the nature of a perceived three-dimensional object, such as a body B, from its two-dimensional image $\pi(B)$. For example, suppose B is a polyhedron with vertices $b_1, b_2, \ldots, b_n$. Given $\pi(b_i)$, and the equations for π, can we find the x, y, z coordinates of b_i? It is not hard to see that if we could calculate the distance of b_i to the pinhole, we could get the desired coordinates. This is not a classical problem in projective geometry, because it is impossible to invert π in this way. But the human eye-brain system seems to do something rather like this all the time! How can the human mind do what is mathematically impossible? One way to explain this is the fact that human beings have two eyes and their cooperation helps confer on us the capability of depth perception (determining the distance from an object to the eye – this is often called stereo vision). A mathematical description of how this could be done for a robot with two cameras (two perspectives) is a standard topic in computer vision textbooks. (An even more interesting, but less studied, question is to explain the interesting fact that people with sight in one eye can carry out some depth perception.)

Linkages and Configuration Spaces

Figure 4 shows a simple planar linkage consisting of line segments hinged at the point where they meet and with a hinge at the base point B. Such a linkage can be regarded as a simple model of a rather specialized robot (specialized because robots normally operate in three dimensions, simple because we represent the links as line segments instead of the more complicated solid bodies they actually are.) Despite the simplification and specialization, the study of such linkages can provide much of the essential intuition and formal mathematics for studying robots.

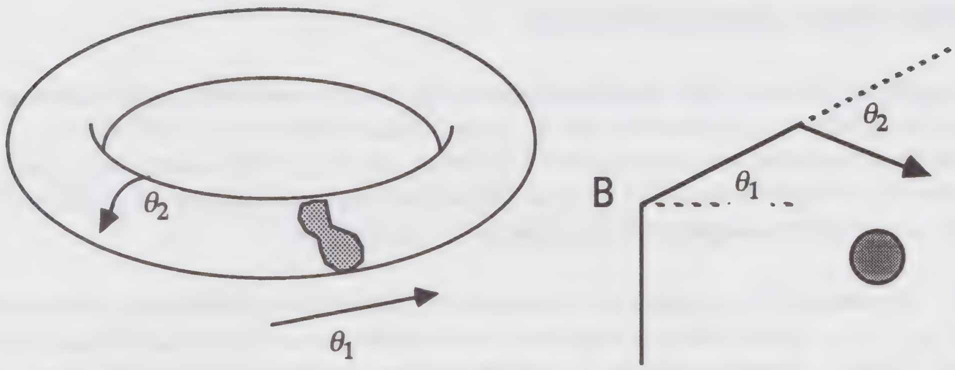


Figure 4.
The torus as a configuration space for a 2 joint planar robot.

An idea of particular importance is the idea of configuration space. Every position of the linkage is determined by the pair of angles [θ_1, θ_2]. This ordered pair of angles is called a configuration and the whole collection of configurations is the configuration space of the linkage. The configuration space for **Figure 4** is a torus since each angle is measured mod 2π. Associated with this linkage, there is a natural mapping from a configuration [θ_1, θ_2] to the point (x, y) where the "end-effector" (tip of the robot) is. This is called the "forward kinematic transformation". For Figure 4, it can be regarded as a mapping from the torus to a portion of the plane. (We discuss the formulas for this mapping in the next section.)

Questions about the linkage and the mapping are often regarded as belonging to topology and analysis. However, many of these questions and their answers can be visualized intuitively and are, consequently good subjects for geometry. For example, what is the range of the forward kinematic transformation? For a given motion of the linkage, what does it look like as a moving point on the torus? If we have "joint limits", i.e. restrictions of the type $a \le \theta_1 \le b$, this makes the configuration space less than the complete torus. Exactly what does the configuration space look like now? For a given point (x, y) in the range of the forward kinematic transformation, how many pre-images does it have?

Obstacle avoidance is an important issue for robots. Suppose we have an obstacle in the workspace of the robot, such as the shaded circle in Figure 4, and no portion of the robot is allowed to touch the obstacle. This rules out certain configurations in the configuration space. In other words, each obstacle in the workspace has, as its counterpart, a configuration space obstacle consisting of all configurations that entail an intersection of some part of the linkage with the obstacle. The shaded blob in the workspace of the robot in Figure 4 might correspond to the shaded patch on the torus.

A motion of a point on the torus (corresponding to a motion of the linkage) must avoid all such configuration space obstacles in order to be safe. It is interesting to have an idea of what configuration space obstacles are like. If the obstacle is a point, what does the corresponding configuration space obstacle look like? In particular, could it disconnect the torus? Could it "go around the torus" in one direction? The key things about these questions are:

1. You don't actually need the apparatus of equations and algebra and calculus to get the answers.
2. Getting the answers involves thinking back and forth from the toroidal configuration space model to the linkage in its workspace – a characteristic mathematical activity.

Calculating Robot Arm Kinematics

Now suppose we want the equations describing the forward kinematic transformation, i.e., we want equations that predict (x, y) based on the θ_i. In our discussion of this, we'll also remove the artificial assumption that the links are line segments. It turns out that, for the questions discussed below, the shape of the links plays no essential role, so there is no harm in having robot links depicted as realistically as we can manage with our sketching capabilities.

So now we assume that a robot arm consists of a series of rigid bodies, called links, which we may denote by $L_0, \ldots, L_n$. L_0 is called the base and, for simplicity, we regard it as fixed (for example, bolted to the ground). There is an end-effector, L_e, which can be a gripper, drill, or other tool, attached to the last link. We'll assume that this is a rigid attachment. However, the other links, L_i, $1 \leq i \leq n$, can move with respect to one another and the description of this movement is typically done using the concepts and language of isometries. We give a brief outline of this next, using the example of the two-dimensional robot shown in **Figure 5**.

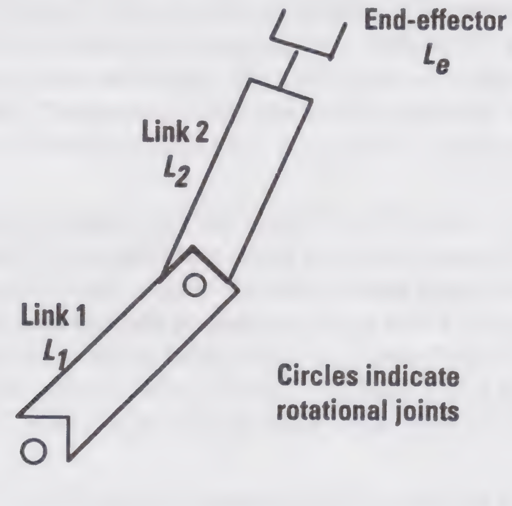


Figure 5.
A two link robot with gripper.

Let's fix attention to a pair of consecutive links L_{i-1} and L_i. For simplicity, we will stick to robots where these two links rotate with respect to one another. In this case, there is a point, called the point of rotation, whose position relative to L_{i-1} is fixed and whose position with respect to L_i is fixed also. However, the relative orientation of L_i with respect to L_{i-1} is not fixed, but is determined by a rotation angle θ_i. In Figure 5, rotational joints are depicted by little circles and the point of rotation, in each case, is the center.

The rotation angle, θ_i, needs to be measured from some reference position of link L_i at which the angle is considered to be 0 (**Figure 6**). This reference position can be chosen in any way that is convenient. Thinking about what is most convenient is best postponed until we describe the placement of local coordinate systems in the various links.

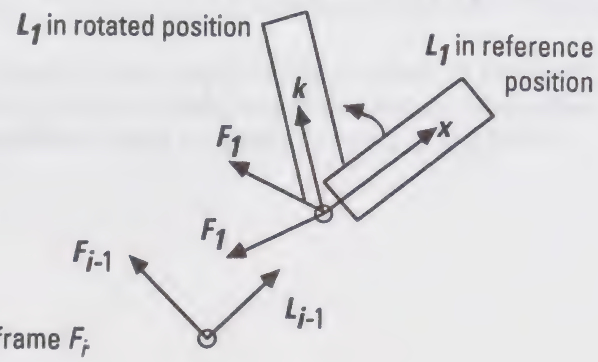


Figure 6.
Two positions of L_i and the corresponding positions of the frame F_i

The whole point of the robot's rotations is to move the end-effector to some position and orientation in space where it can do useful work. To describe this, it is useful to place a coordinate frame at the end-effector in some convenient way. For example, if the end-effector is a gripper, we might take the origin midway between the jaws of the gripper, and the y-axis pointing from one gripper finger to the other.

F_n, the coordinate system for L_n, has its origin is at the point of rotation which L_{n-1} and L_n share. Pick the x-axis to point to the center of the coordinate system F_e. For any other link L_i, install F_i in L_i as follows. Pick the origin of F_i at the point of rotation which L_{i-1} and L_i share. (This choice of origin makes it convenient to represent the rotation of L_i using the reference position of coordinate system F_i.) Pick the x-axis to point to the next center of rotation, i.e., the center of frame F_{i+1}. Notice that as L_i rotates, its coordinate frame F_i rotates with it. The coordinate system for L_0 can be chosen arbitrarily. (**Figure 7.**)

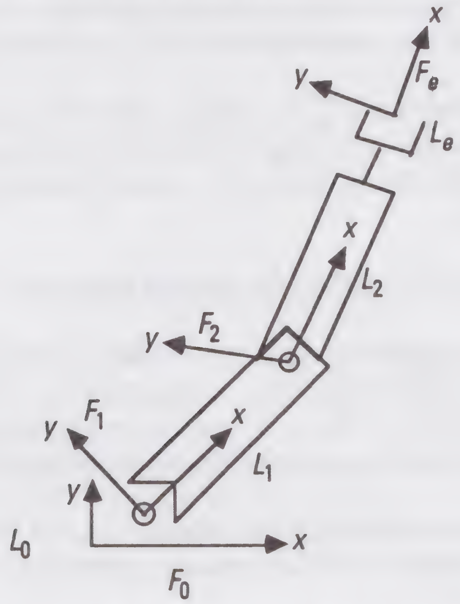
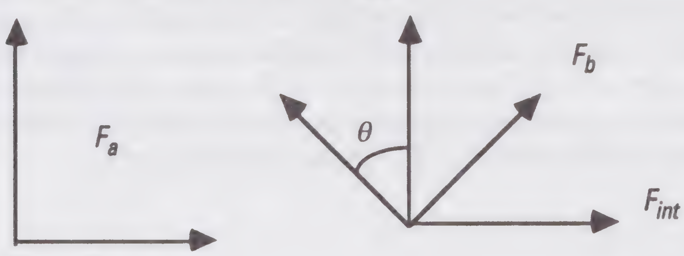


Figure 7.
Coordinate frames for the links of a 2D robot.

Now to mathematics. Suppose we are given two coordinate frames in the plane, F_a and F_b. We may ask: what isometry needs to be applied to F_a to yield F_b ? In our application to the link frames, it will be convenient to regard this isometry as occurring in two steps:
a) a translation of F_a to an intermediate frame, F_{int}, which has the same origin as F_b.
b) a rotation around the frame F_{int} so that the rotated version of F_{int} coincides with F_b.



M_{int} = translation matrix moving F_a to F_{int}

M_{rot} = rotation matrix moving F_{int} to F_b

$M_{ab} = M_{int} \times M_{rot}$

Figure 8.

If we introduce homogeneous coordinates, then the translation and rotation can be represented by 3 x 3 matrices and the composition can be regarded as a 3 x 3 matrix. In this way, we can regard the frame F_b, which we originally think of as a geometric entity – a point for its origin and two directed rays for the axes – as having a numerical representation and a matrix. But, of course, the representation is relative to F_a. If we change to a different F_a, the matrix that represents F_b will be different.

A key mathematical theorem we will use is:

Theorem

Consider three coordinate frames F_a, F_b, F_c and let the matrix representing F_b in F_a be A_{ab} and let the matrix representing F_c in F_b be A_{bc}. Then the matrix representing F_c with respect to F_a is $A_{ac} = A_{ab}A_{bc}$.

Now let's consider the use of this theorem in the case of links L_i and L_{i+1}. There is some known isometry T_i (computed when the robot was built) which transforms F_i to the reference position of F_{i+1}. It is convenient to pick the reference position of L_{i+1} $[\theta_i = 0]$ so that this isometry is a translation. Now suppose L_{i+1} is rotated by θ_i and let $R[\theta_i]$ be the 3 x 3 homogeneous matrix representing rotation by θ_i. This is the isometry that sends the reference position of F_{i+1} to the rotated verstion of F_{i+1}. Then the theorem implies that the rotated version of F_{i+1} is obtained from F_i by the isometry $A_i\,[\theta_i] = T_iR[\theta_i]$.

Now let's consider links L_0, L_1, and L_2 with their frames F_0, F_1, F_2 and transformation matrics $A_1\,[\theta_1]$ $A_2\,[\theta_2]$ as just described. Applying the theorem once again shows that the isometry carrying F_0 to F_2 is $A_1\,[\theta_1]\,A_2\,[\theta_2]$. Finally, let A_e denote the isometry carrying F_2 to F_e. Then the isometry which carries the base frame F_0 to F_e is

$$\mathrm{A}_{\mathrm{total}} = A_1\,[\theta_1]\,A_2\,[\theta_2]A_e.\ \text{ In general, if we have } n \text{ joints, } \mathrm{A}_{\mathrm{total}} = A_1\,[\theta_1] \ldots A_n\,[\theta_n]A_e.$$

The foregoing equation is the matrix version of the forward kinematic transformation (first described above in the section on linkages and configurations). What good is it? There are a number of uses, all of which are based on the fact that it is routine for a robot to be able to measure its joint angles θ_i. Therefore it knows, no matter what position it is in, what the numerical value of matrix $\mathrm{A}_{\mathrm{total}}$ is. Consequently,

1. The robot knows where its end-effector is since the origin of the coordinate frame F_e (expressed in base coordinates) is the last column of the matrix $\mathrm{A}_{\mathrm{total}}$.

2. The robot knows how the end-effector is oriented since the directions (in base coordinates) of the axes of the end-effector frame F_e are given by the initial columns of the matrix $\mathrm{A}_{\mathrm{total}}$.

3. Suppose the end-effector has some kind of sensing apparaus which locates objects. For definiteness, suppose the object is a point which has been perceived by a camera and a distance sensor which are mounted near the end-effector and which move with it. Assume that the point's coordinates in the end effector coordinate system are determined from the sensor readings. (In general, any sensor which is mounted on the end-effector and moves with it can only give end-effector – as opposed to base – coordinates for perceived objects.) But we may need to know the coordinates of the point in the fixed base coordinate system, F_0. The key relationship is:

   Theorem:

   Let X_e and X_0 be the coordinates of a point in end-effector and base coordinate systems respectively. Then

$$X_0 = \mathrm{A}_{\mathrm{total}}X_e$$

Minkowski Addition

Recall that if A and B are two sets in a vector space – say R^2 – then $A + B$ is $\{a + b : a \in A, b \in B\}$. This algebraic concept has interesting geometric content. For simplicity, assume that the origin is in A. Put your finger on this point and then slide A around (i.e., generate a family of translates of A) so that your finger occupies, in turn, all the points of B. The part of the plane swept out by A as it moves is $A + B$ (**Figure 9**).

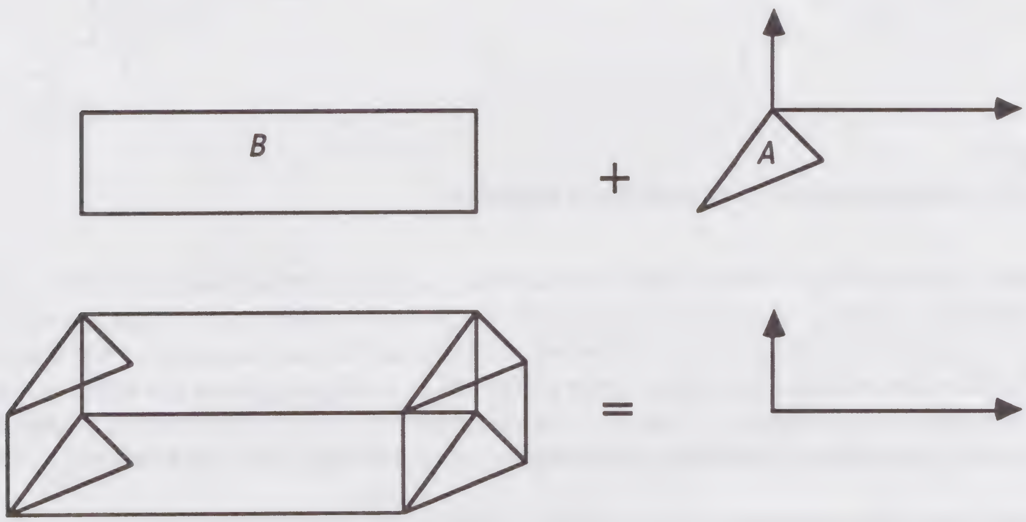


Figure 9.
Minkowski addition of *A* and *B*.

This is not hard to prove and is the fundamental intuition about Minkowski addition. Students can readily construct (by free-hand drawing) sums of convex polygons. For arbitrary convex bodies, one needs the support function in order to construct $A + B$. However, if A is a circle of radius r and B is a polygon, one can still manage with free-hand drawing (**Figure 10**). In this case, $A + B$ consists of all points within r units of B. In robotics lingo, this is a "grown" version of B, or B with a cushion of safety around it.

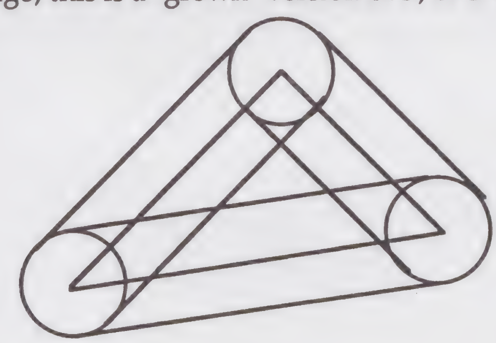


Figure 10.
Minkowski sum of a circle and a triangle.

When a robot must move amidst obstacles without touching them, it is useful to place a suitable safety zone about B. By planning to avoid $A + B$, instead of just planning to avoid B, one creates some insurance against the inevitable errors the robot makes in juding the size, shape and position of B, and errors by the mechanical drive system that moves the robot.

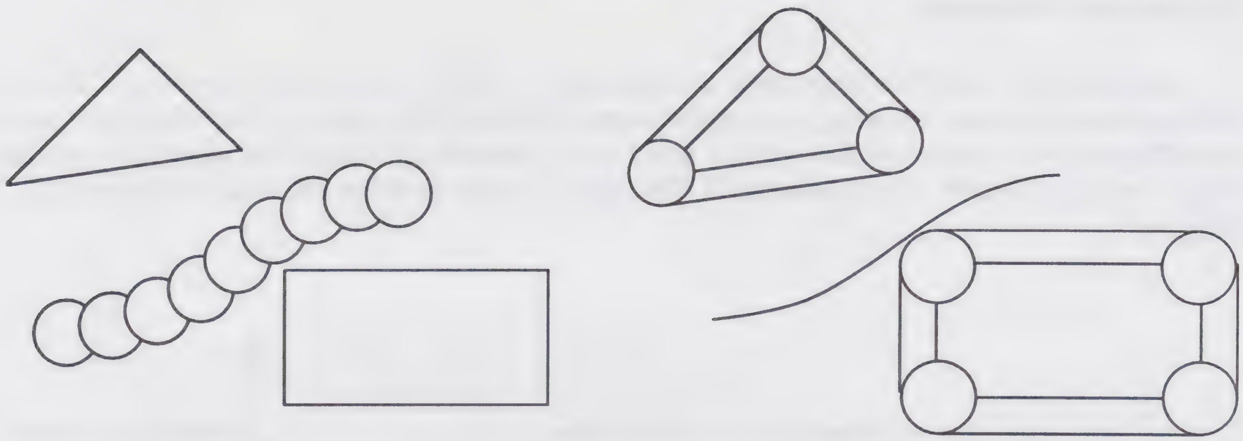


Figure 11.
Reducing a swept area to a curve amid grown obstacles.

Another important use of the Minkowski addition occurs when we have a mobile robot moving about on the floor needing to plan a motion from one point to another, avoiding convex obstacles E_1, E_2, E_3, ..., E_n. For simplicity, imagine that the robot is a cylinder whose footprint on the floor is a circle of radius r. Let the footprints of the obstacles be polygons as in **Figure 11**. As the robot moves, it sweeps out a complicated area and it is hard – both at the level of intuition and at the level of algorithm development – to cope with the question of whether this swept area crosses any obstacle footprint.

The following theorem allows some simplification. Let the robot footprint at time t be $R(t)$ and let the position of its center at time t be $c(t)$. Let C_r be the circle of radius r centered at the origin.

Theorem

$R(t)$ does not intersect any E_i provided $c(t)$ does not intersect any of the grown obstacles $C_r + E_i$.

The import of this theorem, which is easy to prove, is that one can plan a path for a moving point amidst the grown obstacles instead of planning for a moving disk among the original obstacles.

Convex Geometry in Undergraduate Instruction

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A Question and Some Answers

I have been asked to discuss the suitability of convex geometry (the geometry of convex bodies) for inclusion in an undergraduate geometry course. I'll do that, but I want first to consider the following more general question:

What are the desirable characteristics of an undergraduate geometry course?

In proposing some answers to this question, I am thinking of a course that is aimed primarily at fourth year undergraduates who are majoring in mathematics or in some area of science. For students who are preparing to teach mathematics in high school, some of my answers would be different, but I would still maintain (a), (b), and (f).

(a) The course should not consist solely of axiomatic Euclidean geometry.

When courses for prospective high school teachers are included, axiomatic Euclidean geometry is at present the subject most frequently taught in college geometry courses. In many small colleges, it is the only geometry course offered, perhaps because it is thought to be essential for prospective high school teachers and because the available resources do not permit offering two geometry courses. However, despite its long tradition, the axiomatic approach to Euclidean geometry is unlikely to provide much useful information or experience for the math and science majors who are my main concern here. Even for prospective high school teachers, axiomatic Euclidean geometry is far from an optimal choice.

(b) It is more important to study the geometric objects, their structure, and their properties, than to have a coherent axiom scheme.

One should use the most powerful approach rather than the most aesthetic one. Any axiomatic basis should be specifically "geometrical" in nature only if no loss of efficiency results from this. The course should be <u>content</u>-oriented rather than <u>foundation</u>-oriented. It should introduce the student to new geometric experiences, not concentrate on proving almost-obvious statements about familiar objects. It should focus on geometric objects and ideas, not primarily on the underlying logic. This certainly does not exclude proofs, but it does involve approaching geometry somewhat as a branch of physics, the viewpoint being that geometric objects really exist and we want to learn more about them. Experimenting, conjecturing, and proving should all be a part of this learning process.

(c) Emphasis should be given to the use of geometric insight as a way of attacking problems in a variety of areas.

The areas would of course include those that are explicitly "geometrical" in nature, but might also include parts of analysis, algebra, number theory, combinatorics, etc. One picture is indeed worth a thousand words, even if the picture is only a mental one that cannot, because of its high dimensionality, be transferred to a piece of paper or a computer screen.

(d) The material considered should be set in *n*-space, not restricted to the plane or 3-space.

Such a restriction would be a letdown for the students (who by now presumably have some proficiency in multivariate calculus and linear algebra), and would significantly reduce the usefulness of the course. The most powerful aspect of the geometric viewpoint is that, once one has become somewhat accustomed to the formalities involved in thinking about higher-dimensional spaces, many high-dimensional situations can be pretty well understood on the basis of low-dimensional intuition. (There are plenty of counter-examples to this, but the statement still has much validity.)

(e) There should be a unifying theme throughout the course.

It might be the study of certain objects, such as differentiable manifolds or convex bodies. Or it might be some notion, such as invariance or symmetry.

(f) If the subject matter permits, unsolved problems should be mentioned to whet the students' interest.

Undergraduate students are generally much less aware of the research frontiers in mathematics than in other areas of science. Their interest will certainly be heightened if they can be shown that mathematics (and, in particular, geometry) is well supplied with fascinating questions that have not yet been answered.

How does a course on Convex Geometry Meet the Above Criteria?

Some of the subjects that might satisfy the above conditions for a geometry course are differentiable manifolds, geometric topology, algebraic topology, *n*-dimensional projective geometry, and convex geometry. Of course, there are of other possibilities. Not surprisingly, my own favorite candidate is the last one: convex geometry. At least for undergraduates, and at least if the *n*-dimensional theory is to be covered, a course in differentiable manifolds is likely to require so much background in analysis and a course in algebraic topology to require so much background in algebra that the underlying geometric ideas may be hidden from the student. This might not apply to a course in *n*-dimensional projective geometry, but that subject falls short (at least as compared to convex geometry) so far as potential applications are concerned.

With the study of convex geometry (i.e., of convex sets in *n*-space), the above-mentioned desiderata are met in the following ways.

(b) The method that seems best to provide a powerful approach is to start in the setting of a real vector space. It does not matter whether the students have ever seen an axiomatic approach to the real number system. Most of them are willing to assume the basic properties of real numbers as "experimental facts," and proceed to apply them. The essential notions from linear algebra are not extensive: linear and affine subspaces, rays, lines and line segments. Even students who are not completely comfortable with matrices are not at a great disadvantage. The approach through linear algebra (plus a few basic topological notions such as closedness and compactness) is quick and powerful, and leads rapidly to interesting, useful, and even surprising results.

(c) Here I'll mention just three elementary (and yet very important) applications of convex sets in other parts of mathematics. They are ones that I have often included in a junior/senior-level course on the geometry of convex bodies. Of course there are many other possibilities.

(i) In functional analysis, two of the most important tools are separation theorems for convex sets (the basic separation theorem being equivalent to the Hahn-Banach theorem) and extreme-point theorems for convex sets (the Krein-Milman theorem and its relatives). Inclusion of these topics (at least their n-dimensional versions) in a convexity course serves simultaneously to teach the students some interesting geometrical facts and to prepare them for a later course in functional analysis.

(ii) In a very natural way, a number of important results from topology can be brought into a course on convexity. For example, the Euler characteristic can be approached in a combinatorial way based on convex sets. Also, after having defined simplicial subdivisions and proved that a simplex admits subdivisions of arbitrarily small mesh, then after establishing a combinatorial lemma of Ky Fan (taking perhaps one hour), it is possible to prove (in about two hours) such basic and exciting results as Brouwer's fixed-point theorem, the Borsuk-Ulam antipodal mapping theorem, the invariance of domain theorem, the Lebesgue tiling theorem, etc. One can then apply these results in obtaining further properties of convex bodies.

(iii) The study of high-dimensional convex polytopes provides the essential geometric underpinning for the study of linear programming. And convex sets are involved in a large fraction of the modern developments in computational geometry, with their motivation from problems in computer graphics, robotics, etc.

(d) The development is easily carried out in n dimensions, partly because the basic notion is so intuitive and simple. It has often been remarked that many of the methods used in studying the geometry of n-dimensional convex bodies are essentially 2-dimensional or 3-dimensional. This makes motivation and appeal to the intuition very easy, even in n dimensions.

(e) The underlying theme is of course the study of properties of convex sets. This is very pictorial. Its objects are close in spirit to those of Euclidean geometry, but it makes much more contact with modern mathematics.

(f) As far as unsolved problems are concerned, there is truly an embarrassment of riches. As a source of easily stated and intuitively appealing unsolved problems, the study of convex bodies rivals elementary number theory, graph theory, and combinatorics.

What are the Chances of Agreement?

While the general subject of convexity is, by a wide margin, my own first choice of a topic for an undergraduate geometry course, it is fair to suspect me of being prejudiced. (Someone once said about Minkowski that he "was interested in all things convex," and the same is true of me.) Naturally, other mathematicians will have their own pet topics, and I would be amazed if there were any general agreement as to choice of topic. There are three ways out of this dilemma:

(i) Simply don't offer geometry at the undergraduate level;

(ii) Give a single course that attempts to provide at least a taste of several different parts of geometry;

(iii) Offer a regular full-year (or at least two-quarter) undergraduate course in geometry, devoted each year to a single area of geometry, but change the area from year to year, either according to a fixed schedule or according to the interests of prospective instructors.

I strongly prefer (ii) to (i), and with equal strength prefer (ii) to a course that is devoted entirely to axiomatic geometry. However, I believe that (iii) is by far the best choice here. The sort of hodgepodge course that is mentioned under (ii) is fairly common, but in my view it has the serious drawback of never getting to material of significant depth and having little chance to meet the criteria (c) and (f) mentioned above.

Whatever topics are chosen for the geometry course, it is essential that their visual or pictorial aspect not be lost. The geometric paradigm is unparalleled for its ability to represent a complicated mathematical situation with a few strokes of the pen. That is what should be emphasized -- not the analytic aspects, not the algebraic aspects, but the truly geometric aspects!

Geometry and the Imagination

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In the spring semester of 1990, the three of us organized and ran a new course at Princeton, called Geometry and the Imagination, after the famous book by Hilbert and Cohn-Vossen. This course grew out of a series of discussions and experiments with other courses, reflecting our general dissatisfaction with undergraduate mathematics education both for majors and nonmajors.

This was a course for freshmen and sophomores which, in contrast to many if not all of the courses we have taught in the past, was really alive. There were many enthusiastic students who put energy into the subject because it was exciting to them, not because they need it for the exam or some other outside purpose.

We don't mean to draw a hasty conclusion that the course should be copied elsewhere – but we feel that we have learned some important lessons from which many others could also benefit and that should be kept in mind in trying to redesign a basic college course in geometry.

What we present here is a collection of handouts we prepared and distributed during the course. These documents pretty well capture the nature of the course as it grew and evolved. We hope that they will help to show what the course was like, and also give some concrete ideas that you can use if you want to teach a course like it.

You will note that the materials thin out toward the end of the course. This is mainly the result of a protest from a number of members of the class. This protest took place on Princeton's "hug-a-tree day." The students complained that we were generating too much paper, and insisted that we use transparencies instead of handouts. Doubtless we could have prepared the transparencies on the computer, as painstakingly as we had prepared the handouts, but somehow we didn't have the energy for it.

What Led to the Course

There were many factors that led us to believe that radical changes in the curriculum were necessary. The number of math majors has been about 15 per year at Princeton, far too small a number. Undergraduate enrollments in service courses have also been declining. In most mathematics courses, undergraduates are unresponsive during classes: it is very hard to get them to enter into a good discussion of the material. On exams, students exhibit only a weak grasp of concepts. Much of what they learn one semester, they forget the next semester. Somehow, mathematics to them is not a connected whole, but a bag of recipes and facts which they can memorize shortly before an exam so that they measure up to the desired level relative to their fellow students. In accordance with the mood of undergraduates, most of the mathematics faculty are unenthusiastic about teaching: it is a chore to get out of the way.

Another stimulus came from Bill Thurston's experiences as a judge for the Westinghouse Science Talent Search. It seems to be very rare, even among students with the best science educations in the country, to understand the connection between science or mathematics and their reality. For example, few of even the best students understand that the phases of the moon are due to the fact that the moon goes around the earth – most think it is because the shadow of the earth falls on the moon. Few know that an octave represents a factor of two in frequency of sound. Few understand stereoscopic vision or understand how to find the distance to a distant object by triangulation. If people don't even know or make connections on these elementary levels, what use will they ever make of sophisticated mathematics?

Most courses taught for undergraduates are a part of a calculus sequence (including here linear algebra taught as part of multivariable calculus, differential equations, and real analysis). Calculus is a problematic subject to teach these days, because of the exposure many students have in high school to some or all of calculus. The best students place out of the introductory calculus, but few students have really learned it well. It is just hard to treat a subject which students have been through before without either making it boring because of the repetition, or too sketchy to be comprehensible to those who don't already have a good understanding. As the only undergraduate exposure to mathematics available to most students, it seems very drab and unrepresentative.

How It Worked

The original flyer for the course, together with an initial handout on philosophy and organization, explain some of our thinking in planning for it. (You may want to take a look at them now before reading on.) Not everything turned out quite the way we thought it would – except that we didn't really think that everything would all turn out the way we thought it would. Here are some of the aspects of the course that we think were most important.

Group discussions and cooperative learning. One of the key features of our course was the organization of discussions (cooperative learning). We had about 90 students, where we initially planned for 20 or 30. Unlike most other schools, the Princeton math department does not generally teach in large lectures. Calculus, for instance, is taught in classes of size 20 or 30. Rather than split our class into disjoint sections, however, we decided to teach as a team and break during class periods into discussion groups of size about 8 consisting entirely of students.

The students in the class were highly mixed: there were high school students and adult members of the local community, there were students majoring in mathematics and other sciences, and students majoring in art history and humanities.

As a rule, we prepared discussion questions each day. Because of the mixed nature of the class, as well as a matter of principle, the questions needed to be real questions, which make sense without much formal background. Often we gave two or more questions, so that a group could find something interesting to do if one or more of the questions was too hard or easy. We were amazed at how well students engage in these questions, given the chance. It is very important that they have a chance to talk about concepts in their own language, and raise their own concerns. The usual inhibitions evident in discussions led by a professor are gone. Probably an average of 40 minutes of the hour-and-twenty-minute class periods were spent in these discussions. We found that students got so involved that they weren't ready to rush out as soon as the official time was over: we often stretched the classes to more like an hour and a half.

Collegial approach. Another key feature was our team approach to teaching. Teaching tends to be not fun, in a research department such as Princeton's, partly because it is done in isolation from our colleagues. Teaching as a team, rather than in independent sections, we find that we enjoy debating about what we should do next, how we should present it, and who will do it. With three of us leading the course, we had much more time and energy to plan and prepare for each class so that it would work well. It is also very important to discuss afterward what worked and what didn't. We love to hear the sounds of our own voices, and show off our cleverness – it is very important to have our colleagues around to criticize us and help us better to understand the reality of the students.

Active learning. We also think it is important not to go off for very long in pure uninterrupted lecture mode. We interrupted each other liberally, and this seemed to help keep up the interest and attention of students.

The use of physical materials (manipulables) was a very important feature of our course which went along with the de-emphasis of lecturing. Students, as a whole, are not in a position to explore for themselves a mathematical theory as an abstract body of knowledge, since they are not well connected to it. They are much smarter when it comes to discovering patterns connected more directly to them – particularly physical materials.

For example, we handed out mirrors every day during the next two weeks while discussing symmetry, and students also used paper and scissors. Students come to take it for granted that when you ask them a problem, they can often use physical materials or at least their imagination to try to solve it. Models of polyhedra, polydrons, modeling clay and string were staples of class discussions. Students often came up with their own materials to try to help work out what we are talking about.

Less is more. Many courses are jammed with an agenda, which leaves no room for the input of students. There are so many concepts to be covered that any relaxation of the pace in order that students can work out some of the relationships for themselves is impossible.

The result of a course like that tends to be that the material is indeed covered – rather than revealed. Students can cram to memorize recipes for an exam, but they do not absorb it well enough that they ever use it again outside of a narrow context like the one in which it is taught.

When we planned the course, we made a list of lots of topics that we would love to convey to the students. The list was so long that it was obvious to us we could not treat them formally even in two years, let alone a semester. This freed us from any feeling that there was a definite agenda of topics we have to cover. The only hope of communicating the spirit and usefulness of geometry is to aim at teaching students an active approach to investigating geometric problems.

We feel that this paid off. Instead of students returning (on homework and tests) much less than is fed in, students return more.

The down side. Of course there were things that didn't work as well as we hoped, and things we'd do differently.

The discussion groups sometimes splintered into smaller groups, probably because the group size of approximately 8 that we favored was too large. Things might have worked better with groups of 3 or 4.

The special interest groups that we organized kind of fizzled out, except for one or two. Partly this was because the students didn't have specific times when they could be depended on to be able to come, such as they would have had in a laboratory course with lab periods scheduled when they signed up for courses. Consequently, it was hard to arrange to hold these meeting at times when all those who were interested could come. Probably more important was the fact that these groups weren't required, and once the term got into full swing, students found that their time was taken by activities that they really had to do. Students take too many courses.

Another thing we wondered about was whether we should have offered the course pass-fail only, as we did (somewhat to the dissatisfaction of the dean). Our reason for doing this was that we wanted the students to be free to think creatively, without worrying if they're going to get a good grade. This worked, and there was none of the usual "Do I have to know this for the exam?" business. But while there were many students who worked very hard and got a lot out of the course, there were others who worked pretty hard and got pretty much out of the course – but not as much as we felt they could have. Many of these must have been thinking that there was no question that they would pass our course, so their time would be better spent grinding away for A's in their other courses. And some of the students who did work very hard asked themselves, "what's the point of doing all this work when all I'll have to show for it on my transcript is a P?" We don't care much about grades, but we care about the students – even the ones who care about grades.

The geometry room was used a lot by some of the students, and some by a lot of the students, but it never turned into the center of gravity of the course as we had hoped. Of course we had planned on having many fewer students; the room was far too small to serve as a center for so many students. And the math building is off at the edge of campus, so it's not the kind of place that students are likely to drop in. The room was a real center for geometry, and more, but there were as many faculty members and graduate students coming by as there were undergraduates.

Was it a gut? We didn't set out to teach a gut course – quite the opposite. There is nothing wrong with gut courses, but this wasn't supposed to be one of them. We wanted only students with a real flair for mathematics. Well, we definitely got some of those, but we got others as well, who were not as gifted mathematically, but who had a hunger to learn math that wasn't being satisfied by the other math courses they were offered. We think we were able to satisfy that hunger, and yet we felt that we had plenty to offer even the most gifted students.

The Verdict

The course was hailed as a success. Considering the amazing amounts of time and effort we put into it, it would have been sad indeed if that had not been the case. As they say, all educational experiments succeed, if only because the teachers get all fired up about teaching. We thought there was more to it than this. We thought we were actually doing something different that was really better.

Reference

Thurston, William P. 1990. "Mathematical Education." *Notices of the American Mathematical Society*. Providence, RI: American Mathematical Society; 37; 7: 844–850.

Mathematics 199: Geometry and the Imagination
John Conway, Peter Doyle and Bill Thurston

Princeton is an active and exciting world center for mathematics, yet few undergraduates see anything connected with this activity. What most undergraduates see from the mathematics department, if anything, is a sequence of calculus courses which are often dreary to both student and instructor.

This course is an experimental offering on the freshman and sophomore menu of something besides ten varieties of calculus: something better representing the richness, diversity, connectedness, depth, and pleasure of mathematics. The course is named after a famous book by Hilbert and Cohn-Vossen which will provide a good deal of the reading. "Geometry" is taken in a broad sense, as used by mathematicians, to include such fields as topology and differential geometry as well as more classical geometry. "Imagination," an essential part of mathematics, means not only the facility which is imaginative, but also the facility which calls to mind and manipulates mental images. One aim of the course is to develop the imagination.

Several additional books will be used for reading or reference, including Abbot's *Flatland*, Dewdeney's *The Plainverse*, Coxeter's *Geometry*, and Weeks' *The Three-dimensional Shape of Space*.

The course is intended for students who are prepared to work hard and imaginatively, but who might or might not have a strong background in mathematics. The mathematical content will be high, but we will try to make it as independent of prior background as possible. Calculus is not a prerequisite.

This course is offered Pass/D/Fail only, but it is not a course for students who hope to get by with minimum effort. Homework will be assigned every class, due the following class. Assignments will involve thinking and writing, not just grinding through formulas. There will be a strong emphasis on readings, projects, and discussions rather than lectures. All students will be expected to get involved in discussions, within class and without. We will have a graduate and an undergraduate assistant, in addition to the instructors. A "Geometry Room" in the heart of the mathematics department will be reserved for students in the course and others who are interested. The room will accrete mathematical models, materials for building models, references related to geometry, a computer workstation, questions, responses, and (most important) people. There will be a major final project rather than a final exam.

Philosophy and Organization

Philosophy

"Lego blocks for jocks," read the headline in the *Yale Daily News,* in their article about Mathematics 199 at Princeton. "Even though Princeton undergraduates attend the nation's second best university" the article began (referring to the most recent *US New and World Report* survey rating Yale first and Princeton second), "next term they will get to play with Legos for credit."

This course is *not* based on the principle that most students are dumb, incapable of learning "real" mathematics, and therefore little can be expected from them. On the contrary: we believe that students are far smarter and have much more potential that most of them reveal in traditionally-taught mathematics courses.

What do you think? Have mathematics courses you have taken brought out the best in you? Have you put your best into them?

The spirit of mathematics is not captured by spending three hours solving 20 look-alike homework problems. Mathematics is thinking, comparing, analyzing, inventing, and understanding. The main point is not quantity or speed – the main point is quality. The goal is to reach a more complete and a better understanding.

We will use materials such as Tinkertoys, Legos, mirrors, scissors, and tissue paper not because we think you are too stupid to solve algebraic equations and differential equations, but because we think you are too smart to omit thinking and reasoning from your courses.

We have arranged this course to be P/D/F not because we expect worse than A-quality work, but because we expect better. Too often with grades, students judge their work against that of other students in a course, get a feel for what level of energy is necessary to get an A, B, or whatever grade they are aiming for – and then do that much work but no more. Students in a group look at each other and all lower their educational sights together, just as people in a group generally walk slower than people walking individually. Ultimately, only you are responsible for your own education, and you will have the high quality learning you deserve only by taking charge and doing it yourself.

Some of you may have enrolled in this course because you thought it would be a gut. Gut courses have a valid place in the curriculum: they enable you to take a relaxed attitude some of the time, so you can really concentrate and devote yourself to other things. This course, however, is not a gut: this is a course where we expect you to really concentrate and dig in. If your other courses are arduous, we suggest you either switch some of the others to gut courses so you can concentrate on Mathematics 199, or else switch out of Mathematics 199 to something that will take less time (e.g., Mathematics 211, which has some of the same spirit.)

We are very enthusiastic about this course, and we have many plans to facilitate your taking charge and learning. You won't need a heavy formal background for the course. What you do need is a commitment of time and energy.

Organization

Scheduled meetings

The core of the course is the officially scheduled meeting, Tuesdays and Thursdays at 10:30. You are expected to attend all scheduled meetings. With our large enrollment, we are trying a novel form of organization, in order to enable everyone to be engaged in discussions while at the same time preserving the unity of the course. We will break frequently into small groups of about 8-12 people for discussions of various topics. You will form these groups in place, moving around as necessary.

Each group will have a moderator and a reporter.

The main job of the moderators is not to contribute points to the discussion, but to make sure that *everyone* is involved: that everyone is being heard, everyone is listening, that the discussion is not dominated by one or two people, that everyone understands what is going on, and that the group sticks to the subject and really digs in.

The main job of the reporters is not to contribute points, but to make sure that they understand and write down the key points and interesting ideas in the discussion.

After a suitable time, we will ask for reports to the entire class. This will not consist of formal reports, but it will be a summary discussion among the reporters and teachers, with occasional contributions from others.

Groups, reporters, and moderators will stay the same during each class period. We encourage the groups to remain the same from one period to another, but reporters and moderators should rotate.

Texts

The required texts for the course are:
Hilbert and Cohn-Vossen, *Geometry and the Imagination*; Weeks, *The Shape of Space*; Abbott, *Flatland*; Dewdeney, *The Plainverse*; Peterson, *The Mathematical Tourist*.

All are available at the University Store. By mistake, the Peterson book was ordered in hardcover instead of paperback; it has been reordered in paperback. You can save $7 by waiting for the paperback to come in.

In addition, we recommend: Coxeter, *Introduction to Geometry*. A limited number of copies are available now at the University Store. These are in hardcover, and the price is a hefty $65. Paperback copies priced at around $30 should be coming in at the end of March.

Other materials

We will be doing a lot of constructions during class. Beginning this Thursday (February 8), you should bring with you to class each time: scissors, tape, ruler, compass, sharp pencils, plain white paper. It would be a capital idea to bring extras to rent to your classmates.

Journals

Each person in the course will keep a journal or lab book for the course. Assignments will be given at each class meeting for the journal, but the journal is also for independent questions, ideas, and projects, including ideas and information for your major project. They are not for class notes, but for work outside of class. You are encouraged to cooperate with each other in working on anything in the course, but what you put in your journal should be you. If it is something that has emerged from work with other people, write down who you have worked with. Ideas that come from other people should be given proper attribution. If you have referred to sources other that the texts for the course, cite them.

Journal entries should be legible and *readable*. Neatness counts; grammar counts; spelling counts; above all, *exposition* counts. If you are presenting the solution to a problem, explain what the problem is. If you are giving an argument, explain what the point is before you launch into it. What you should aim for is something that could to communicate to a friend, a parent, or a teacher a coherent idea of what you have been thinking and doing in the course.

Please don't put rough drafts in your journal; work out what you are going to write beforehand, then copy or paste it in.

The one and only officially approved notebook for you journal is the Dennison "75 sheet" (152 page), 11.75" by 9.25" Computation Notebook #43-648, which is sold in the basement of the University Store. These notebooks cost $9 or so, but we think they're worth it. These notebooks are large enough for good sized drawings, and there is plenty of room to paste in standard 11" by 8.5" sheets, so that you can conveniently do your writing with a word processor, if you like. These notebooks have a serious, a substantial, and yet not a somber air to them, and we hope you will be inspired to write in such a way as to be worthy of the notebook you are writing in. Accept no substitutes!

Sadly, the University Store currently has on hand only 50 or so of these notebooks. More should be coming by the end of the week. In the meantime, those of you who can't get hold of a notebook should keep your journals on separate sheets, which you can paste or copy in later.

We will collect journals to review every other Thursday, starting the second Thursday of the term (February 15). However, you are expected to keep your journal up-to-date, and in particular to complete each assignment before the next class meeting. Each entry in your journal should be dated – don't go back and fill in later, but if you have more ideas about something you wrote previously, refer back to it by page number and date.

To encourage you to adhere to these guidelines, and also as a way of taking attendance, we will date-stamp your journal at the beginning of every class, starting the second Tuesday of the term (February 13). So: DON'T FORGET YOUR JOURNAL!

You are responsible for keeping a copy of everything that goes into your journal. How you do this is up to you: You can make carbon or xerox copies, or extra printouts if you are using a word processor. Just make sure that you have copies of everything. That way, if you should lose your journal, things will not go quite so badly for you.

Constructions

Geometry lends itself to constructions and models, and we will expect everyone to be engaged in model-making. There will be minor constructions that may take only half an hour and that everyone does, but we will also expect larger constructions that may take several evenings or a weekend. We keep a long list of suggested constructions, and we will supply more details as needed.

Final project

We will not have a final exam for the course, but in its place, you will undertake a major project. The major project may be a term paper investigating more deeply some topic we touch on lightly in class. Alternatively, it might be based on a major model project, or it might be a computer-based project.

Mid-term project

In addition to the final project, a project will be due at mid-term. This can be a preliminary version of your final project, or something completely different.

Special interest groups

We will organize a number of special interest groups on special topics. Possible topics include mathematical education, mathematical drawing, computer graphics, dynamical systems and chaos, differential geometry. The idea is to give you extra information and ideas for you projects, as well as to allow you to focus more closely on topics you are particularly interested in. These groups will meet approximately once a week, and continue as long as seems appropriate.

Geometry room

Room 311 on the third floor of Fine Hall will serve as a special work and play room for this course. This is where you can go to find the famed Lego blocks, as well as other toys, games, models, displays, and construction materials. Copies of handouts and other written materials of interest to students in the course will be kept here as well. It should also serve as a place to go if you want to talk to other students in the course, or to one of the teachers. Our current plan is to have the room open from 1:30 to 3:30 PM Monday through Thursday, beginning right away.

Reserve books

A large number of mathematically interesting books will be placed on reserve in Fine Hall Library. You can browse through these books to look for ideas for projects, or just for fun. Look for these books in the library's xerox room, rather than in the bookcases where reserve books are usually kept.

Special evening lectures

We will have four special evening lectures during the term, aimed at the general public as well as the students in the course. The first of these will take place on the evening of Thursday, February 15 at 7:30 PM in Jadwin A-10. The speaker will be Scott Kim, a calligrapher, typographer, and computer artist from Palo Alto, California. The title of the talk is "How to invent your own mathematics."

How to pass this course

If you come to class, do your best on the assignments, engage fully in talking and thinking and imagining and building models, write thoughtfully in your journal, and do substantial midterm and final projects, you will pass the course. You don't have to win a race with the math majors.

A. Symmetry and Reversals

These questions are topics for discussion. You can concentrate on one or both of these questions, depending how the discussion goes – this isn't a test.

1. What does "symmetry" mean? Can you give a clear definition or explanation for this concept? What are some good examples of objects or of patterns illustrating different kinds of symmetry? How can you differentiate among different kinds of symmetry?

2. When you look at your image in a mirror, it has a right and left interchanged. If you raise your right hand, the mirror person raises the left. How can the mirror know left-right from up-down: why doesn't your mirror image have top and bottom reversed instead? Imagine a neon sign on a glass store-front, with a mirror behind it. does the image of the sign read left-to-right or right-to-left? How is an image reversed in a camera? What about an image in your retina?

February 6: Assignments

Discussion questions (By 2/8/90)

Write down further thoughts, clarifications, or dissenting views you have on the questions which were handed out for class discussion.

Snowflakes and Paper dolls (By 2/8/90)

Make some symmetrical patterns by folding paper, cutting, and unfolding. How many patterns of folding did you learn in childhood? Can you come up with any other patterns of folding and cutting which give symmetrical figures? Tape or glue some good examples into your journal, and describe them in words. How many repeats are there in each one?

Other symmetrical patterns (By 2/8/90)

Draw some symmetrical patterns which cannot be constructed by folding. Why can't they?

Flatland and the Plainverse (By 2/15/90)

Read these two books and be prepared to discuss them.

B: Symmetry and Reflections

Symmetry is a geometric rhyme
-- Rachel Findley

Form into groups, not too large, and choose *new* moderators and reporters.

Everyone in the group has the right and duty to understand what is going on, and the right and duty to contribute to the discussion.

The primary job of the moderator is to make sure that this happens: that everybody participates and nobody dominates.

The primary job of the reporter is to take notes so as to be able to summarize and represent *the whole group's* ideas and conclusions.

Everyone should read the questions below. Then discuss topics 0 and 1 while circulating mirrors through your group. After everyone has had a chance to experiment with the mirrors, go on to topic 2.

2. When you hold two mirrors at right angles and look into them, you can see three images of yourself, if you are close enough. Adjust the angle to make the middle image as good as you can. Try putting small objects or designs next to the mirrors.
Which of the images are right-handed and which are left-handed? At what other angles can you hold the mirrors and still see an exact number of exact images? What are the possibilities for the number of exact images you can obtain?

0. Old business. Are there questions left from Tuesday's discussion, or from the homework, which want to be discussed? Choose at least one pattern of folding and cutting, and volunteer someone in the group to make it while the group discussion goes on to share with the whole class.

1. Classify the letters of the alphabet as to symmetry. That is, describe which letters are transformed to themselves by various transformations, and which letters transform into other letters.
What words, phrases, or sentences can you construct which transform into themselves by vertical reflection, horizontal reflection, or 180° rotation of the paper?

More constructions of symmetry (Assignment due 2/13/90)

Experiment with the constructions below. Put the best examples into your journal, with captions which describe and explain what is going on.

1. **Triangular folding patterns.** There are three triangular folding patterns for symmetrical designs in the plane. They are based on triangles whose angles are 45°, 45°, 90°, or 60°, 60°, 60°, or 30°, 60°, 90°. You can lay out triangles in these shapes with the help of a ruler and a compass. Fold paper into these patterns, cut, and unfold to make symmetrical designs.

   Note that by folding, you are subdividing each triangle into similar subtriangles. By choosing different folds, you can vary the number of subtriangles. Try experimenting with the 30°, 60°, 90° triangle to see how many different ways you can divide it into subtriangles.

2. **Conical patterns.** Many rotationally-symmetric designs, like the twin blades of a food processor, cannot be made by folding and cutting. However, they can be formed by wrapping paper into a conical shape.

   First, fold a sheet of paper in half, and then unfold. Cut along fold to the center of the paper. Now wrap the paper into a conical shape, so that the cut edge lines up with the uncut half of the fold. Continue wrapping, so that the two cut edges line up and the original sheet of paper wraps two full turns around a cone.

   Now cut out any pattern you like from the cone. Unwrap and lay it out flat. The resulting pattern should have two-fold rotational symmetry.

   Try other examples of this technique, and also try experimenting with rolling the paper more than twice around a cone.

3. **Cylindrical patterns.** Similarly, it is possible to make repeating designs on strips. If you roll a strip of paper into a cylindrical shape, cut it, and unroll it, you should get a repeating pattern on the edge. Try it.

4. **Möbius patterns.** A Möbius band is formed by taking a strip of paper, and joining one end to the other with a twist so that the left edge of the strip continues to the right.

   Make or round up a strip of paper which is long compared to its width (perhaps made from ribbon, computer paper, adding-machine rolls, or formed by joining several shorter strips together end-to-end). Coil it around several times around in a Möbius band pattern. Cut out a pattern along the edge of the Möbius band, and unroll.

5. **Others.** Can you come up with any other creative ideas for forming symmetrical patterns?

Special Interest Groups

During the regular meetings of the course, we will discuss a variety of topics, but there is not time to go into any single topic in great depth. This week, we begin forming special interest groups to delve deeper into particular topics. These groups are for you, at no extra cost.

We envision groups of between five and twenty students, ideally about ten, to facilitate more direct interaction between teachers and students. You can participate in as many or as few of these groups as you like. Your course projects will probably be much easier if you take part in at least one, but this is not required. We will not hold groups without your interest and enthusiasm.

We have many ideas for possible special topics, but there is a limit to how many you or we can sustain. We have listed some of our ideas below, but we are open to other ideas. Probably only a few of them will actually fly. The accompanying sheet asks for some information from you, including special topics you are interested in. We will schedule organizational meetings for topics with enough interest, combining related topics when it seems appropriate.

There is a blank for your own ideas. The best way to guarantee that a group is actually formed is for you to organize a group of students. We will keep a sign-up book outside the geometry room (311 Fine Hall).

Possible topics for special interest groups

1. **Computer modeling of geometry.** With the right tools, training, and equipment, it is possible to make pictures of some kinds of geometric drawings much more quickly and easily than by hand, and to model three-dimensional objects more quickly and easily than with physical materials. Some kinds of computer modeling are impossible or very difficult to do at all with physical models.

   There are several possibilities for tools, training and equipment, depending on demand.

   a. We can arrange for accounts and lessons on high-quality graphics computers (Irises) with excellent three-dimensional capability – probably using the Iris classroom in the CS building and with the help of a computer graphics expert from CS.

   b. We can arrange for accounts and lessons on mathematics department computers using Mathematica, a powerful symbolic mathematics program. The computers are Suns, Irises, and a Next.

   c. We can use University Macintoshes, with their ease of use, highly-developed drawing programs, and relatively easy languages such as True Basic.

   We will try to accommodate varying levels of previous experience with computers, from none to vast.

2. **Three-dimensional solids.** Since the days of Plato and Archimedes, a great and rich lore has developed, with many beautiful and interesting examples. What three-dimensional analogue of the pentagram (five-pointed star) can be constructed? What is the theory of geodesic domes?

3. **General model construction.** We would get together to share models, and techniques for making models, inspired by a variety of geometric topics.

4. **Four-dimensional geometry.** Do you have a mental image of a four-dimensional cube? It is not as hard as you might think to learn to imagine and manipulate four-dimensional geometric figures: in your head, or symbolically with numbers and equations. There are some truly remarkable and beautiful objects which exist only in four dimensions.

5. **Non-Euclidean geometry.** For many centuries people tried to prove the parallel postulate of Euclid either from the other axioms, or from truths which were more self-evident. In the early nineteenth century, mathematicians' thinking was transformed by the discovery of a kind of geometry, otherwise very similar to Euclidean geometry, where the parallel postulate is false. This kind of geometry, usually called hyperbolic geometry, is fun, interesting, and useful.

6. **Chaos and order (Dynamical systems).** Why can astronomers predict the phase of the moon for centuries, but meteorologists can't predict the weather for more than a week or two? These are both instances of dynamical systems. Dynamical systems, whether the rules they follow are simple or complex, sometimes exhibit very orderly behavior and sometimes very chaotic behavior.

7. **Fractal geometry.** How long is the coast of Maine? The answer depends on the length of your measure. If you trace out the coast in steps of ten miles, you answer will be much shorter than if you measure in steps of one mile. That answer, in turn, is much longer than if you measure in steps of .1 mile … and so on down to steps of an inch or so. This is an example of fractal geometry: an object which reveals additional complexity on each smaller scale. Other examples are trees, lungs, mountains, the moonscape … many examples are all around us. The mathematical theory of fractal geometry had its origins about 100 years ago, but has come much more into the public consciousness in recent years with the advent of computer-generated fractal images.

8. **Mathematical education.** How do people learn mathematics? What is the main substance of what they learn: facts of concepts? Do standardized tests help or hinder our system of mathematical education? Are paper-and-pencil computations becoming obsolete, with the widening use of calculators? Do students perform more poorly in the US than in the other major industrialized countries because they watch more television, because they have rotten teachers, because they don't take enough mathematics courses, because the culture does not have a high regard for mathematics, or because the measure of performance is at fault? Why do fewer minorities and few women learn as much mathematics?

9. **The geometry of flexible surfaces (Differential geometry).** Why can't you flatten half an orange peel onto the table? When you bend a sheet of paper, why are there always straight lines along its surface?

   There is a beautiful mathematical theory of the geometry of surfaces which bend without stretching. The art of shaping clothes to fit is a practical side to the same theory.

10. **Topology.** Topology is the mathematical theory of geometric shapes which can bend and stretch, but without tearing or gluing. It can be used to answer questions such as whether or not a loop of string forms a knot, or whether it can be unknotted without undoing the loop. Is it possible to lay a loop of string on the floors and stairways of Fine Hall so that it does not cross itself, but it forms a knot? The topology of surfaces is also quite interesting.

11. **Symmetry.** This would be a continuing investigation, in greater depth, of ideas from the first two weeks of the course. Why are there exactly seven possible patterns of strip symmetry (as in bolts of cloth, and borders for many designs), and 17 possible patterns of full two-dimensional symmetry (as in wallpaper)? Possible symmetry patterns can be understood by studying how they roll up or fold up. Three-dimensional patterns, as in crystallography, are also quite interesting.

12. **Projective geometry and perspective drawing.** The mathematical theory of projective geometry grew out of the development of techniques for perspective drawing by mathematically-inclined artists. Projective geometry has to do with the surprisingly interesting constructions which can be made with a ruler alone. How can you make a map of the earth which shows the shortest distance between two points as a straight line?

13. **Mathematical drawing.** Why does a drawing of a torus have a smile? This is explained by singularity or catastrophe theory. There are drawings in most mathematics books which don't look right because they are not mathematically correct. There are a number of ideas and techniques (besides perspective, above) which can help enable amateurs to make good sketches of three-dimensional figures.

14. **Inversive geometry.** The geometry of circles is more symmetrical and beautiful than most people are aware. How can you make a map of the earth which shows everything with the proper shapes? How can you turn the plane inside out in such a way as to send every circle to a circle?

15. **The geometry of the world and the universe.** Geometry derives from a Greek word meaning measurement of the world.

    How can you construct a pyramid whose base is accurately square? How can you measure the distance between Fine Hall tower and the graduate tower, without leaving Fine Hall? How can you measure the diameter of the earth, without leaving Princeton, using tools and materials that are readily available? How can you tell the time of day from the stars, or from the sun? If you see a half-moon rising on the night of March 21, what time is it?

16. **Spherical and solid geometry.** These are topics which are used to be included in many high school geometry courses 20 years ago, but have since been dropped. They have many applications to the world around us.

17. **Analytic geometry, elementary algebraic geometry.** This is the study of curves and surfaces defined by equations. This subject is rich and highly developed both in examples and in theory.

18. **Ruler and compass constructions.** What figures can be constructed with ruler and compass, obeying the classical rules? Why is it possible to construct regular pentagons and regular 17-gons with ruler and compass, while it is impossible to construct a regular 7-gon and impossible to trisect and angle?

19. **Mechanical linkages.** How can you make a mechanical linkage (as with Tinkertoys or Erector Sets or Meccano) to draw a straight line, trisect an angle, or replace the chain on a bicycle? There is a rich and powerful theory of mechanical linkages which tells how to do these and many other things.

Information Sheet

Name: ____
Address: ____
____
____
___Freshman___Sophomore___Junior___Senior___Other
Major_________________Telephone__________
Check once if you are interested, twice if you are enthusiastic about the following topics for special interest groups.

___1. Computer modeling of geometry (___a___b___c)
___2. Three-dimensional solids
___3. General model construction
___4. Four-dimensional geometry
___5. Non-Euclidean geometry
___6. Chaos and order (Dynamical systems)
___7. Fractal geometry
___8. Mathematical education
___9. The geometry of flexible surfaces (differential geometry)
___10. Topology
___11. Symmetry
___12. Projective geometry and perspective drawing
___13. Mathematical drawing
___14. Inversive geometry
___15. The geometry of the world and the universe
___16. Spherical and solid geometry
___17. Analytic geometry, elementary algebraic geometry
___18. Ruler and compass constructions
___19. Mechanical linkages
___Other (Please explain)

What times are best for you?

Assignment due 2/15/90

Questions 1-4 below are phrased in terms of space, but you will probably want to think about and discuss the analogous questions for the plane, first.

1. Tell what it means to reflect space through a plane; a line; a point.
2. Show that reflecting through a *flat* (this is shorthand for "a plane, a line or a point") is an isometry, that is, it preserves distances.
3. Discuss the handedness of images under reflection in a flat.
4. Discuss what happens when you compose reflections through two different planes, i.e., when you reflect in one plane and then another.
5. If an object in the plane is symmetrical about two different lines that meet at an angle of 72°, what other lines is it symmetrical about? What if the lines meet at 135°?

Strip symmetry

Determine the fundamental domains for the following strips (from *Islamic Art*, pp. 106-109), cut them out, and glue them up.

Assignment Due February 20, 1990

0. Finish reading *Flatland* and *The Plainverse*, if you haven't already done so.

1. What happens when you glue triangles, squares, pentagons, hexagons, or heptagons together so that at every vertex you have the same number of polygons?

2. Can you make an annulus out of equilateral triangles? A Möbius strip? A torus? How many triangles do you need?

Comments on journals

General remarks

The quality of the journals was variable. Many people did as we asked and took their journal entries seriously, with pleasing results. Some did not. We are worried about the people who just slapped something into their journals and handed them in. We understand that in some cases there were start-up problems in obtaining journals and figuring out what the system is, but we hope to see everyone up to speed on their journals the next time we collect them.

It is very important that *everyone* taking the course be really engaged in thinking about it, so as not to drag other people down. This is not a question of being fast of slow, but of trying hard to think and to communicate. There are many layers of the material we are discussing. Those of you with more sophisticated mathematical backgrounds should dig into deeper layers, not just coast along because you already know the upper layers (for instance, if you understand the terminology, you might ask and discuss what are the finite subgroups of the orthogonal groups O(2) and O(3).) Those of you who are puzzled by something in class (for instance, stamp symmetry vs. true symmetry) should write about your puzzlement. *Your journal should show what you are thinking,* on whatever level that is.

Responses to all the assignments should go in the journal, along with other questions and thoughts you have in connection with the course. It will help you develop your midterm and final projects to write down your independent thoughts. **You should write complete sentences, paragraphs, and complete short essays.** When you respond to the questions in an assignment, write enough so that someone reading it can understand it without an assignment sheet. You can write down your thoughts on process, as well as substance. For instance, if you are confused about some concept, write down clearly what you are confused about; or if your group discussions seem to be floundering, describe what the trouble is.

Class notes should not go directly in the journal. If you are paying attention in class, you don't have time to write a clear and self-contained journal entry. If you want to take notes during class, do it separately. If you want to record and discuss in your journal what happened in class, do it later.

Specific remarks

Symmetry and reflections

Many people had good discussions of the definition of symmetry, but there were also some common questions: in particular, the distinction between stamp symmetry and true symmetry. A pattern has *stamp symmetry* if it is composed of multiple copies of a single building block (or *fundamental domain*). For a pattern to have *true symmetry,* not only must it be made of multiple copies of a single building block, but all the blocks must have the same relation to the other blocks.

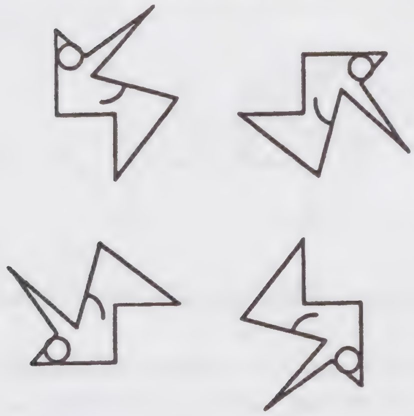


Figure True Symmetry. *This pattern has true symmetry, because each of the four building blocks has the same relation to the others.*

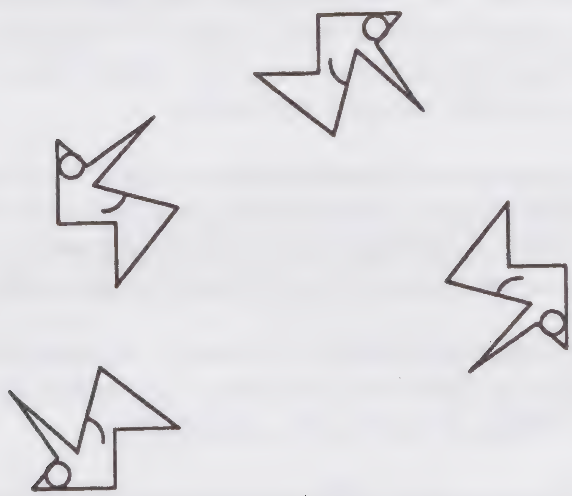


Figure Stamp Symmetry. *This pattern is not symmetric, even though it is made of four identical parts. The parts do not have the same relation to each other, so this is only stamp symmetry.*

In class, the definition of symmetry was given in terms of an *isometry*, which is a transformation which preserves all lengths. A pattern in the plane is *symmetrical* if there is an isometry of the plane to itself (other than the identity) which preserves the pattern. These isometries send one fundamental domain to another. If it is a true fundamental domain, then the same transformation, extended arbitrarily far in every direction, must send every copy of the fundamental domain to another copy of the fundamental domain. Some patterns have finite symmetry, while other patterns have infinite symmetry. (At least one student objected to this use of the term "symmetry," and preferred to call it "repetition." This may be right, but at least for the purposes of the class, we will use "symmetry" to include infinite symmetry.)

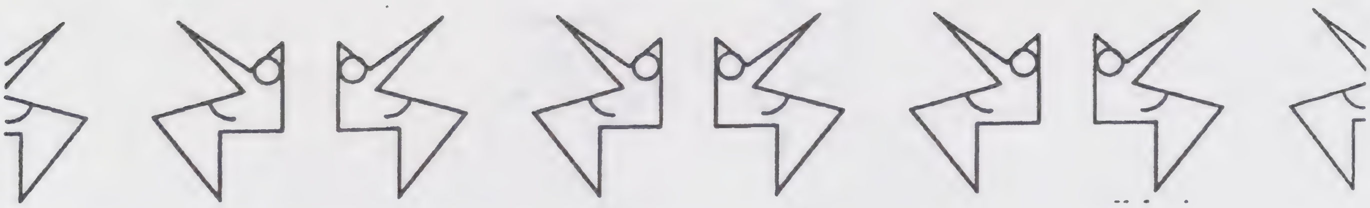


Figure Barber Shop. *The pattern seen in the multiple reflections in two parallel mirrors shows infinite symmetry.*

No paper cutout can show all of an infinite pattern. In talking about the paper cutouts as symmetric, we are really referring to the infinite pattern. In making actual cutouts, it is important to have many repetitions, enough to give the impression that it is part of something infinite.

Triangle folding patterns. Some people did not construct the right patterns for the triangular folding. In these patterns (for instance, with 30° - 60° - 90° triangles), each adjacent triangle is obtained by reflecting the triangle through one of its sides. In folding these patterns, it is important to fold everything into a single copy of the fundamental domain, and then cut. Don't make different folds at different times. It is also important to fold to make as many triangles as possible.

Cylindrical patterns, Möbius patterns and conical patterns. All three of these patterns are made by rolling paper, rather than folding it. One common mistake was that after forming a cone, cylinder of Möbius band, people would flatten it, creating folds. A flattened cone can be created much more simply by simply folding paper. A flattened cylinder gives the same pattern as the paper dolls.

In order to demonstrate a certain pattern of symmetry, it is helpful to make the basic repeating unit asymmetrical. Some people cut out symmetrical units – this makes the result look more symmetrical than the pattern of folding or rolling, disguising the actual pattern.

Many people had difficulty getting Möbius patterns to come out nicely. One of the common mistakes was having too few repeats, often only one or two. An infinite symmetric pattern needs several repeats to reveal itself. One method is to make a Möbius template fairly short so that there will be enough repeats, and roll it across the page. The other method is to make a Möbius band by coiling a strip of paper several times with a half-twist each time, then cutting.

Three-dimensional solids

The analogue of a polygon in three dimensions is a figure whose faces are polygons: it is called a *polyhedron* (plural *polyhedra*).

A *regular* n-gon is one whose sides are all equal and angles are all equal. It has $2n$-fold symmetry, since it can be reflected about any line through its center and any vertex or the midpoint of any side.

How does the notion of a regular polygon generalize to three dimensions? There is more than one possibility.

Form into groups, each with a moderator and a reporter. Remember that the moderators and reporters should rotate. During the discussion, your group should construct at least one three-dimensional polyhedron, for use with the class as a whole.

1. Discuss your experiences with the assignment. What regular three-dimensional polyhedra did you construct? How many faces do they have? How many edges and how many vertices?

2. What definition or definitions would you propose for a three-dimensional regular polyhedron?

3. Is it possible to make a polyhedron so that each face is the same, but so that you would not call it regular?

4. Is it possible to make a polyhedron such that each vertex is the same, and each face is a regular polygon, but you would not call it regular?

5. Can you design flat patterns for polyhedra in a single piece? Try some examples.

Assignment due 2/22/90

1. Discuss your experiences and results from gluing together polygons, from the assignment and from the class discussion. Sketch pictures as best you can, and try your hand at designing flat patterns (called *nets*) for the regular polyhedra. (This will probably be hard at first, but the exercise is important for building up your three-dimensional imagination.)

2. Write an essay on thoughts triggered by *Flatland* and *The Plainverse*. It doesn't have to be long.

22 February 1990

What are the social views of the author of Flatland?

Why have mathematicians been so fascinated by Flatland?

What happens if you suspend a cube by one of its vertices and lower it down through Flatland? What about other solids? What about lowering a four-dimensional cube down through Spaceland?

Assignment due 27 February 1990

How much harder would it be to build a Pyramid twice as high as the one the neighboring Pharaoh just built? What if you and your minions were twice as tall as your neighbors?

15. Course projects

There are no tests in this course; in their place we are asking for major projects. If you have not already done so, it is time to begin planning and organizing your project. In an earlier handout, we spoke of a midterm project. This will not be a separate project, but rather a proposal and a preliminary study for your final project.

Within the next week and a half, you should discuss an idea for a project with one of the instructors, and write down a brief plan in your journal. One of us is often available in the geometry room, particularly during the scheduled hours Monday-Friday from 1:30 to 4:00. If you want to check whether someone is there outside these hours, the telephone number for the geometry room is ALUMINUM (258-6468). The special interest groups are a source of ideas and advice on projects. Our aim in this course is to help you be at your best, and we will do what we can to help you do a good project. Come talk to us when guidance would be helpful.

We are flexible as to the nature of the project, so long as it is something significant. We want projects that you and we will be proud to show others. Term papers are a possibility for projects, but we expect that most projects will be something physical or practical together with accompanying documentation and analysis.

As a minimum standard for projects, we have in mind a certain scale whose units are Keplers and Riemanns. An adequate project could consist of 17 Keplers worth of models, plus 7 Riemanns worth of documentation, or it could consist a 17-Riemann paper plus 7 Keplers worth of models or drawings.

Some ideas for projects

Below are some of our ideas for things that could make good projects. This list is not systematic. It is intended just to help you get started thinking what you would like to do, and to indicate better the kinds of things we have in mind. The best project will be something coming from you – perhaps related to an interest or hobby you have, perhaps something completely new to you which you find intriguing. There is no set formula. We hope to see connections you make for yourself between geometry and the world.
17 Keplers of models, with accompanying document.

1. Write a computer program that allows the user to select one of the 17 planar symmetry groups, start doodling, and see the pattern replicate, as in Escher's drawings.

2. Write a similar program for drawing tilings of the hyperbolic plane, using one or two of the possible hyperbolic symmetry groups.

3. Make sets of tiles which exhibit various kinds of symmetry and which tile the plane in various symmetrical patterns.

4. Write a computer program that replicates three-dimensional objects according to a three-dimensional pattern, as in the tetrahedron, octahedron, and icosahedron.

5. Construct kaleidoscopes for tetrahedral, octahedral and icosahedral symmetry.

6. Construct a four-mirror kaleidoscope, giving a three-dimensional pattern of repeating symmetry.

7. The Archimedean solids whose faces are regular polygons (but not necessarily all the same) such that every vertex is symmetric with every other vertex. Make models of the Archimedean solids.

8. Write a computer program for visualizing four-dimensional space.

9. Make stick models of the regular four-dimensional solids.

10. Make models of three-dimensional cross-sections of regular four-dimensional solids.

11. Design and implement three-dimensional tetris.

12. Make models of the regular star polyhedra (Kepler-Poinsot polyhedron).

13. Infinite Euclidean polyhedra.

14. Hyperbolic polyhedra.

15. Make a (possible computational) orrery.

16. Design and make a sundial.

17. Astrolabe (Like a primitive sextant).

18. Calendars: perpetual, lunar, eclipse.

19. Cubic surface with 27 lines.

20. Make a convincing model showing how a torus can be filled with circular circles in four different ways.

21. Turning the sphere inside out.

22. Stereographic lamp.

23. Flexible polyhedra.

24. Models of ruled surfaces.

25. Models of the projective plane.

Some questions of topology

Divide into groups as usual, and choose a reporter and a group leader (by rotation) for each group. Read through the questions on the front and back of this sheet. Your group as a team should try to answer them, probably by dividing into subteams of two to work on different questions, and later exchanging what you have learned. In the ensuing class discussion, your reporter can call on others from the group, if desired.

1. What is the largest number of points on a sphere such that each pair of points can be connected by a path which doesn't pass through any third point nor any other path? What about on a torus (the surface of a doughnut?)

2. A mathematical knot is a closed curve in space. Two knots are equivalent if one can be adjusted to agree with the other one, by moving it continuously in such a way that it does not intersect itself.

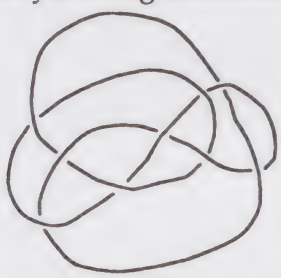
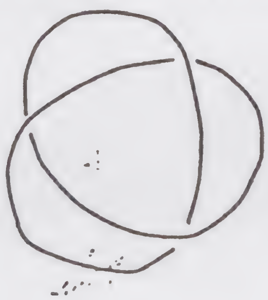


Figure knot. *This is a drawing of a knot with nine crossings. How many inequivalent knots can you find with five or fewer crossings? (Remember to think of loops of string, rather than pieces of string with two ends.)*

3. Is it possible to have a surface whose boundary is a knot? Which knots is it possible to obtain in this way?



4. Can you construct a gluing diagram to obtain a two-holed torus by gluing edges of a polygon together?

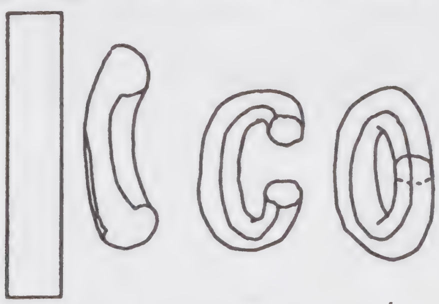


Figure Torus. *If opposite sides of a flexible rectangle are glued to each other, as indicated, the result is a torus. In the reverse direction, it is possible to cut a torus along two intersecting curves so that the result is a rectangle.*

Figure Two-Holed Torus. *Generalize the construction which works for the torus to a construction for the two-hole torus, illustrated above, by finding curves so that if you cut it open along these curves, you can flatten out the surface to obtain a polygon. If you succeed, try to do it with as few sides for the polygon as you can.*

17. Assignment due 2/27/90

Remember that journals will be collected again on Thursday.

0. Follow-up on the class discussion concerning topology.
1. Brainstorm about possible ideas for your project. At this point, you don't need to be selective, but put down lots of ideas which you could conceivably work out.
2. Start reading *The Shape of Space*, by Jeff Weeks. Read at least the first two chapters.

1 March 1990
18. The Euler Number

Today the discussion will center on the Euler number $V - E + F$, where V, E, and F denote the numbers of vertices, edges, and faces.

1. Determine the Euler numbers of the five Platonic solids, and of as many of the numbered objects that have been distributed as you reasonably can.

2. The diagram below shows three houses, each connected up to three utilities. Can you rearrange the connections so that they don't cross? How?

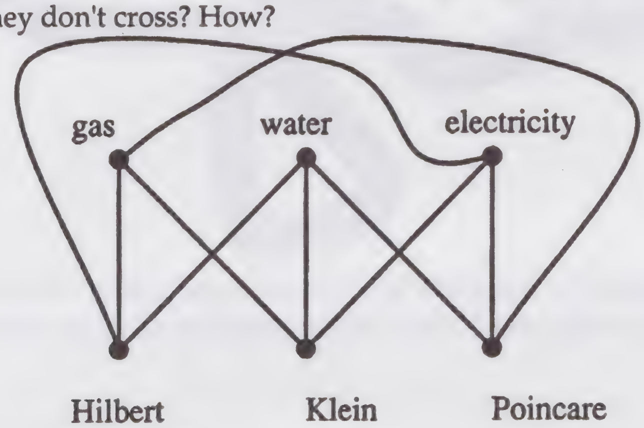


Figure Gas. *The gas-water-electricity problem.*

1 March 1990
19. Assignment due 6 March 1990

1. Continue reading *The Shape of Space*, at least through Chapter 6.

2. Determine the quotient orbifolds for the Escher patterns that have been distributed. How does the answer change if black figures are treated as indistinguishable from white figures?

6 March 1990
20. Assignment due 8 March 1990

Your brother-in-law brings you his bicycle chain with kinks in it, as indicated below, and asks you to help him get the kinks out. Draw a sequence of diagrams indicating what you would do. Describe what you are doing in words. Can you visualize the process with your eyes closed?

Figure Kink. *This diagram represents a chain that has kinks, but is not twisted. Thus we see one side of each link of the chain, except for links that are obscured because another part of the chain lies above them.*

What if the chain looks like this or this or this?

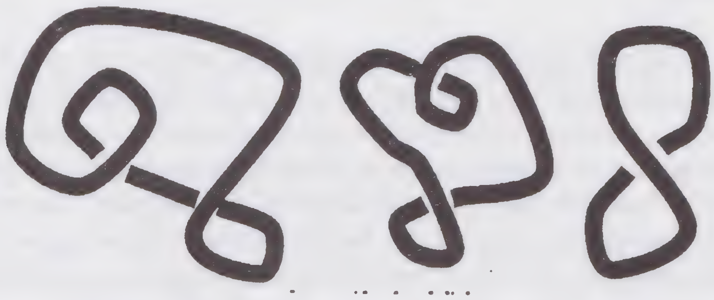


Figure Kinkier. *Again, we see only one side of each link.*

Is there any simple way to tell at a glance if a given configuration can be untangled? What about if you have a miraculous power to change an over-crossing to an under-crossing, as indicated below?

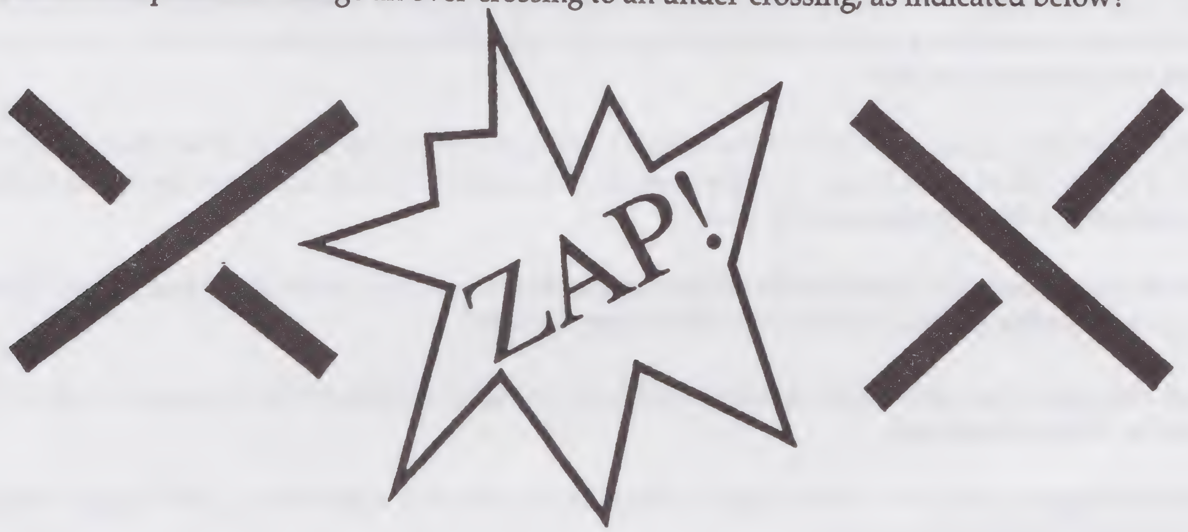


Figure Zap. *Zapping changes an overcrossing to an undercrossing.*

21. Exercises in Imagining

How do you imagine geometric figures in your head? Most people talk about their three-dimensional imagination as "visualization," but that isn't exactly right. The image you form in your head is more conceptual than a picture – you locate things in more of a three-dimensional model than in a picture. In fact, it is not easy to go from a mental image to a two-dimensional visual picture. Three-dimensional mental images are connected with your visual sense, but they are also connected with your sense of place and motion. In forming an image, it often helps to imagine moving around it, or tracing it out with your hands. Geometric imagery is not just something that either you are born with or you are not. Like any other skill, it is something that needs to be developed with practice.

Below are some images to practice with. Some are two-dimensional, some are three-dimensional. Some are easy, some are hard, but not necessarily in numerical order. Work through these exercises in pairs. Evoke the images by talking about them, not by drawing them. It will probably help to close your eyes, although sometimes gestures and drawings in the air will help. Skip around to try to find exercises that are the right level for you.

1. Picture your first name, and read off the letters backwards. If you can't see your whole name at once, do it by groups of three letters. Try the same for your partner's name, and for a few other words. Make sure to do it by sight, not by sound.

2. Cut off each corner of a square, as far as the midpoints of the edges. What shape is left over? How can you re-assemble the four corners to make another square?

3. Mark the sides of an equilateral triangle into thirds. Cut off each corner of the triangle, as far as the marks. What do you get?

4. Take two squares. Place the second square centered over the first square but at a 45° angle. What is the intersection of the two squares?

5. Mark the sides of a square into thirds, and cut off each of its corners back to the marks. What does it look like?

6. How many edges does a cube have?

7. Take a wire frame which forms the edges of a cube. Trace out a closed path which goes exactly once through each corner.

8. Take a 3 x 4 rectangle array of dots in the plane, and connect the dots vertically and horizontally. How many squares are enclosed?

9. Find a closed path along the edges of the diagram above which visits each vertex exactly once? Can you do it for a 3 x 3 array of dots?

10. How many different colors are required to color the faces of a cube so that no two adjacent faces have the same color?

11. A tetrahedron is a pyramid with a triangular base. How many faces does it have? How many edges? How many vertices?

12. Rest a tetrahedron on its base, and cut it halfway up. What shape is the smaller piece? What shapes are the faces of the larger pieces?

13. Rest a tetrahedron so that it is balanced on one edge, and slice it horizontally halfway between its lower edge and its highest edge. What shape is the slice?

14. Cut off the corners of an equilateral triangle as far as the midpoints of its edges. What is left over?

15. Cut off the corners of a tetrahedron as far as the midpoints of the edges. What shape is left over?

16. You see the silhouette of a cube, viewed from the corner. What does it look like?

17. How many colors are required to color the faces of an octahedron so that faces which share an edge have different colors?

18. Imagine a wire is shaped to go up one inch, right one inch, back one inch, up one inch, right one inch,back one inch, ... What does it look like, viewed from different perspectives?

19. The game of tetris has pieces whose shapes are all the possible ways that four squares can be glued together along edges. Left-handed and right-handed forms are distinguished. What are the shapes, and how many are there?

20. Someone is designing a three-dimensional tetris, and wants to use all possible shapes formed by gluing four cubes together. What are the shapes, and how many are there?

21. An octahedron is the shape formed by gluing together equilateral triangles, four to a vertex. Balance it on a corner, and slice it halfway up. What shape is the slice?

22. Rest an octahedron on a face, so that another face is on top. Slice it halfway up. What shape is the slice?

23. Take a 3 x 3 x 3 array of dots in space, and connect them by edges up-and-down, left-and-right, and forward-and-back. Can you find a closed path which visits every dot but one exactly once? Every dot?

24. Do the same for a 4 x 4 x 4 array of dots, finding a closed path that visits every dot exactly once.

25. What three-dimensional solid has circular profile viewed from above, a square profile viewed from the front, and a triangular profile viewed from the side? Do these three profiles determine the three-dimensional shape?

26. Find a path through edges of the dodecahedron which visits each vertex exactly once.

6 March 1990
22. Gas, Water, and Electricity

Why can't you hook three houses A, B, C up to three utilities X, Y, Z without getting some of the lines crossed, as in the attempt pictured here?

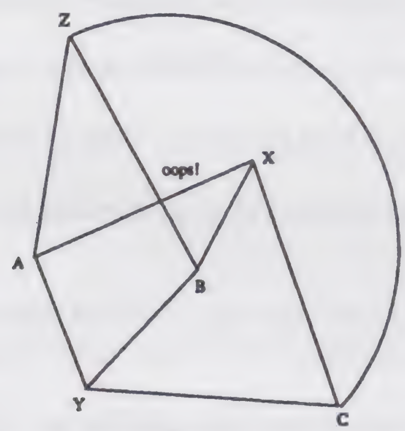


Figure Oops. *A gas leak results when the water company cuts through the gas line!*

Well, you can do it on the torus, because you can start with seven points hooked up each to each, and erase one of the points and a bunch of the lines.

If you could do it in the plane then you could do it on the sphere, and vice versa, by wrapping or unwrapping the picture under stereographic projection, so let's see why you can't do it on the sphere. The proof is by contradiction. If you succeed in hooking everything up without crossing lines. Then you get a cell division of the sphere with $V = 6$, $E = 9$, and therefore $F = 5$ since by Euler's formula $V - E + F = 2$. Each face has at least four edges, because (illegible – Ed.) But 5 faces with 4 edges apiece makes for at least 20 "edges of faces," and as there are 9 edges, each of which contributes exactly 2 "edges of faces," there should only be 18 "edges of faces," a contradiction.

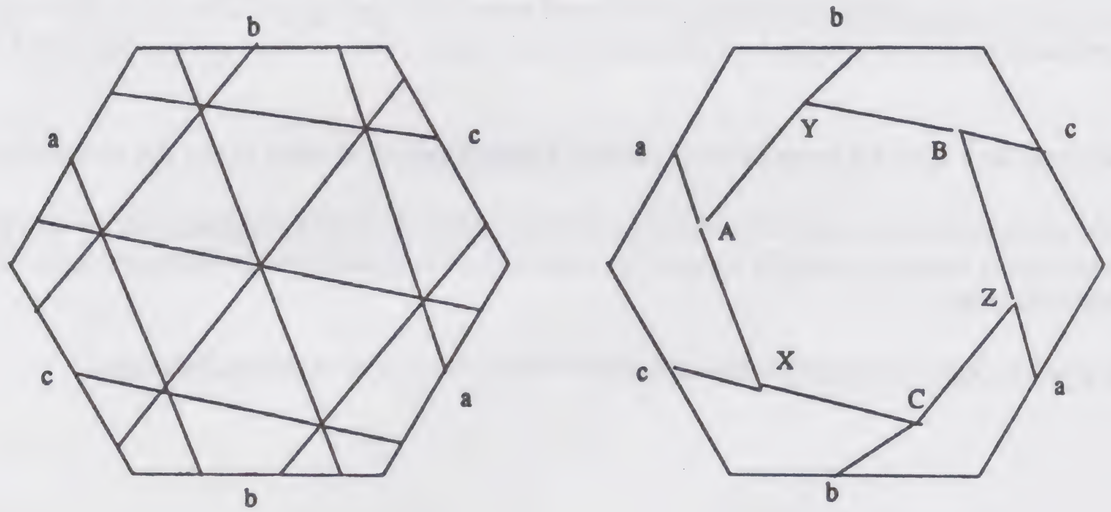


Figure Seven Dormitories on a Torus. *Glue opposite sides of the hexagon together to get a torus with seven dormitories where each pair of dormitories is connected by a path not meeting any other path. On the right, A, B, and C are connected to X, Y, and Z.*

23. Assignment due March 13

1. Find at least two examples of mistakes in figures in mathematics books. Almost any book, such as a multivariable calculus book, will do. Describe in words what the figure is trying to depict, explain why the figure is wrong, and make at least a crude sketch of the correct picture.

2. Read through page 148 in *The Shape of Space.*

Discussion: Stereographic Projection

1. Draw a stereographic projection of a cube, corner down.

2. For the four-dimensional analogues of the tetrahedron, cube and octahedron, count
   V = number of vertices
   E = number of edges
   F = number of faces
   C = number of three-dimensional cells.

3. Which of the three-dimensional sphere, torus, projective space are orientable?

4. Are they different?

Assignment: due Thursday March 15

Midterm Project Reports

As the final assignment before spring break, and in place of a midterm, you should make a progress report/study for your project. Tell as clearly and specifically as you can what you are doing: give references and describe the materials you plan to use. The report goes in your journal, which will be collected on Thursday.

Readings

Stereographic projection is discussed on pp. 248-255 in Hilbert and Cohn-Vossen. Make sure you understand it.
The Mathematical Tourist will be the reading for spring break.

The Thrackle Problem

A thrackle is a doodle in the plane consisting of some special curves called *paths* between special points called *spots*. There are a finite number of paths and spots. Here are the rules:

1. Each path must end in two distinct spots and may not pass through any other spot.
2. Any two paths must have just one point in common. This point must be either an end point of each, or an inner point of each at which they must cross.
3. So no two paths may touch, as in the last figure below:

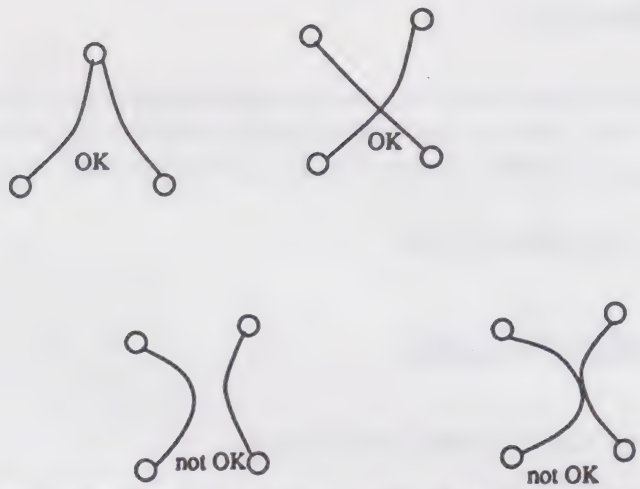


Figure Thrackle Definition. *The top two figures are valid thrackles. The bottom two are not.*

Question. Can there be more paths than spots? John Conway will pay one thousand dollars for the first correct solution.

Subsidiary question. You might also like to consider thrackles on other surfaces, such as the torus. What is the largest number of paths there can be in a thrackle on a torus with *n* spots?

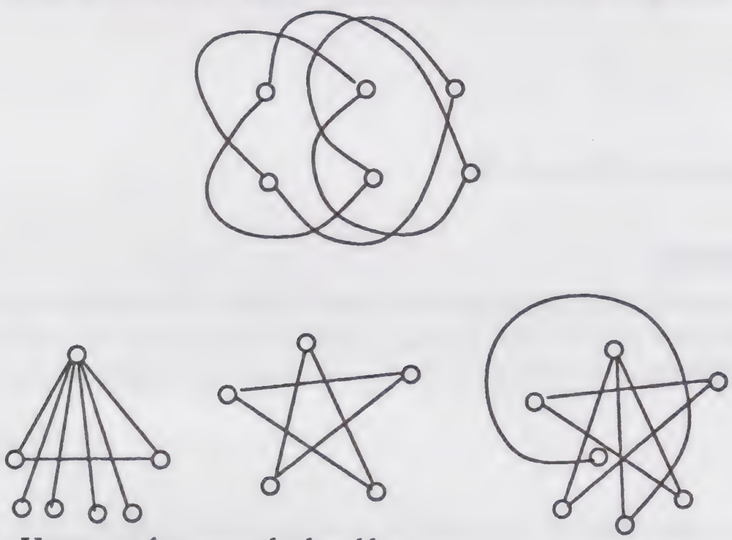


Figure Some Thrackles. *Here are a few example thrackles.*

15 March 1990
27. Discussion

1. Glue opposite edges of an octagon together "without twisting." How many vertices and edges does the resulting surface have? What is its Euler number? What is it? Do the same for the decagon.

2. Glue opposite faces of a dodecahedron together by twisting through one-tenth of a revolution, clockwise in every case. How many vertices, edges, and faces does the resulting object have?

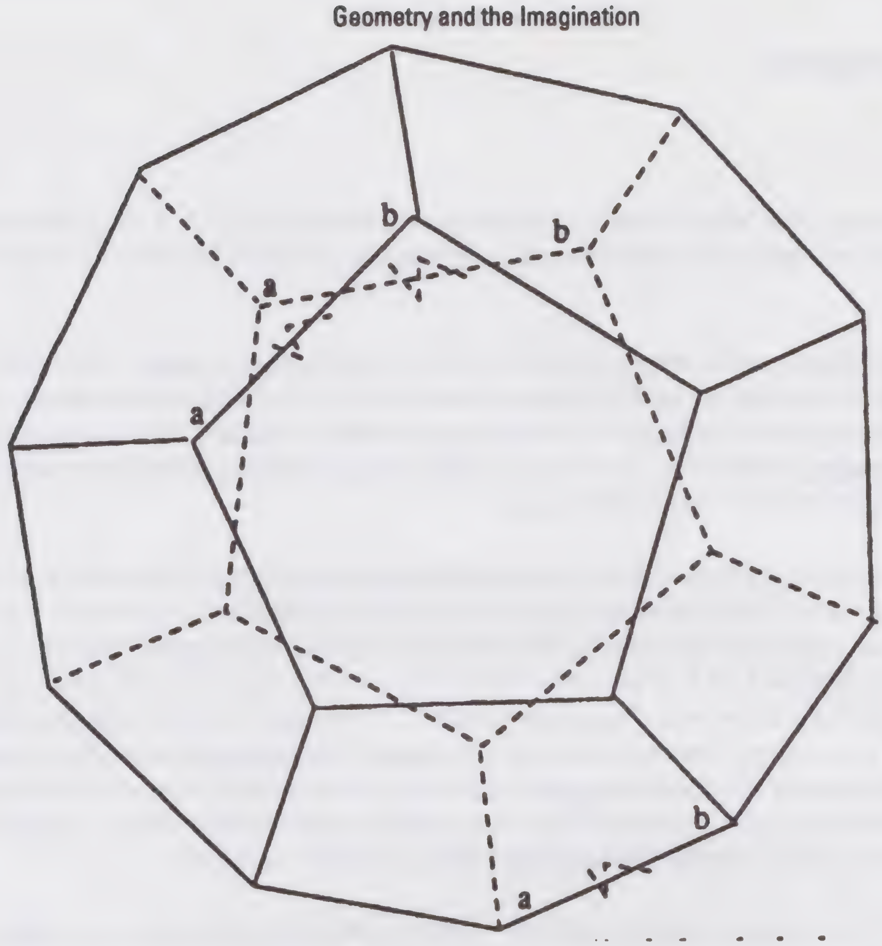


Figure Poincaré Dodecahedral Space. *Poincaré's dodecahedral space is obtained by gluing faces of a dodecahedron. The three edges with squiggles are glued together. Label the rest of the diagram.*

28. Midterm Adjustments

Midterm Progress

In the beginning of the term, we said that this course would be P/D/F only, not because we think you are not good enough to do A-level work, but because we think you can do better than A-level work.

We are very pleased, on the whole, with how the course has been going. The style of organization is an experiment for us, and we had and still have many uncertainties about it. What we are excited about is something sorely lacking in the usual mathematics courses we have taught: a sense that students are really engaging with ideas, that they are discussing, thinking about, and writing about mathematics for themselves and in their own ways.

Because of our renewed enthusiasm and belief in students, we have awarded a fair number of F's and D's for midterm grades. These are based mostly on work as evidenced in journals. It is a basic course requirement that we see your journal every two weeks: anyone who didn't give us their journal at midterm automatically received an F. If you received a D or an F for the midterm grade, don't take this as signaling lowered expectations for you. This should signal to you that we think you are able, and you deserve, to put much more effort into the work for this course. The final grades will be based on your *cumulative* effort in the course. The midterm grade will not be averaged in to form your final grade, so it is not something you have to fight to overcome; it is merely our assessment of your progress so far, based on what we have seen in your journal. We hope that all final grades will be P.

Many students have been putting real effort into the course and into their journals. We think there is a lot they, and *you*, have to gain by putting in effort both during class and outside of class.

We are trying to make this course both enjoyable and intellectually profitable. To protect the course and the students who are really engaged in it, we feel it is important to uphold high standards for work in the course.

Journals

For the remainder of the semester, we will collect journal every second Tuesday, beginning a week from today (April 3), and return them on Thursday. You will have them every weekend, giving you more time with your journal.

27 March 1990
29. Discussion

1. You stand an arm's length away from a plate-glass window, holding in your hand a green Vis-a-Vis overhead projector pen. Outside, a friend holds a perfect square. Keeping very still, you close one eye and carefully sketch the outline of the square on the glass. What are the possible shapes of the polygon you draw? What if your friend is holding a sphere? What if you are crouching inside a plexiglass sphere – with airholes, like a "gerbil ball" – with your open eye exactly at the center of the sphere?

2. You hold a mirror at arm's length, and draw the outline of your face on the glass with your trusty green Vis-a-Vis. How big what you draw to your honest-to-goodness face?

3. Sherlock Holmes claimed that he could tell the direction a bicycle had been traveling by examining the tire tracks. Discuss. In which story does this claim appear?

4. A doughnut is dipped in a trough of coffee as indicated in the diagram below. Describe the "coffee line."

Figure Doughnut. *This figure depicts a doughnut seen in cross-section, floating in coffee so that one disk of the cross section lies just below coffee-level, and the other just above it.*

27 March 1990
30. Assignment due 29 March 1990

Read section 1 and 2 of Chapter 1 in *Geometry and the Imagination.*

1. Use a conventional lamp-with-lampshade (or a reasonable facsimile) to cast elliptical, parabolic, and hyperbolic shadows on your wall. How does the way you hold the lamp determine which figure results?

2. Draw an idealized picture of a perfectly straight train track whose rails meet "at infinity." How can you make sure that the cross-ties appear equally spaced?

31. Conic Sections

Break up into groups and discuss the reading (section 1 and 2 of *Geometry and the Imagination*) in your own words. In particular,

1. Why is the tangent to the circle perpendicular to its radius?

2. Draw some ellipses and some hyperbolas (as best you can) using thread.

3. Why do the two threads from a point on the ellipse to the foci make equal angles with the tangent? What about for the hyperbola?

4. Why is the cross-section of a cylinder an ellipse (that is, why does it have the same shape as one of the thread constructions)?

5. If a plane intersects a cone in a bounded curve, why is the curve an ellipse?

6. Why do circles, viewed from an angle, appear as ellipses?

7. Can a circle viewed from an angle appear as a hyperbola?

32. Assignment due 3 April 3 1990

1. Finish reading chapter 1, at least up to the appendix. Write comments on the reading in your journal.

2. Draw, using a thread construction, a family of confocal ellipses and hyperbolas.

3. Make an elliptical shadow on a wall, and trace it on paper. Locate the center and the two foci.

4. Using the ellipse you have drawn, select six points a, b, c, d, e, f in order around the ellipse. Let

$$\begin{gathered} ae, bf \text{ meet in } p, \\ ad, cf \text{ meet in } q, \\ bd, cd \text{ meet in } r. \end{gathered}$$

Check that p, q, and r are in a straight line.

3 April 1990
33. Assignment Due 5 April 1990

Read sections 15, 16, and 17 in Chapter 3 of *Geometry and the Imagination*.

5 April 1990
34. Projective Transformations

1. Draw four points marked A, B, C, D on one sheet of paper, and another set of four point a, b, c, d on another sheet of paper. Try to imagine the projective transformation that takes the first set of points to the second.

2. Now mark some more points E, F, G, ... on the first sheet and locate their images on the second sheet.

3. Draw a circle on the first plane and draw its image on the second plane.

35. Assignment Due 4/10/90
Mathematical Eduction

Write an essay about mathematical education, based as much as possible on your own experiences and those of your friends. The accompanying essay by Thurston may help stimulate some of your thoughts.

Here are some of the issues which might come up:

Is there too little discipline in the US on getting people to drill on learning their "math facts?" Why are there so many fewer females in mathematics and mathematically-based professions than males? Why are there so few mathematics majors around the country and at Princeton today? Why are there so few blacks and Hispanics in mathematics and other related professions? Do different people have such different cognitive styles for learning mathematics that monolithic lectures are inevitably unsuited to all but a few?

Is the main trouble with mathematics education too much television? Poorly trained teachers? Drugs? Lack of interest by students, in the belief that mathematics is being outmoded by advancing technology? Too much emphasis on standardized tests and memorization? Too little money for the schools? Not enough time for teachers to think and plan for changes? Too large class sizes? Not enough enthusiasm or caring by teachers?

Should calculators and computers be used in a massive way in mathematics courses?

Has mathematics become too formal and removed from the real world? Would many the problems be cured by greater use of "manipulables" in mathematics classes, and more emphasis on how mathematics fits in with real-world problems and phenomena?

Assignment Due 4/12/90

Draw and label configurations dual to the following:

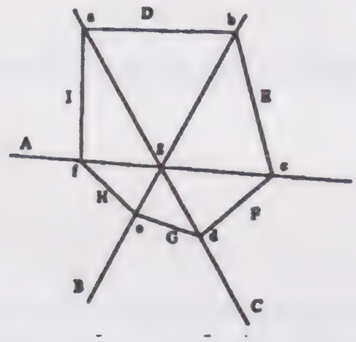


Figure Hexagon. *A hexagon whose diagonals intersect at a point.*

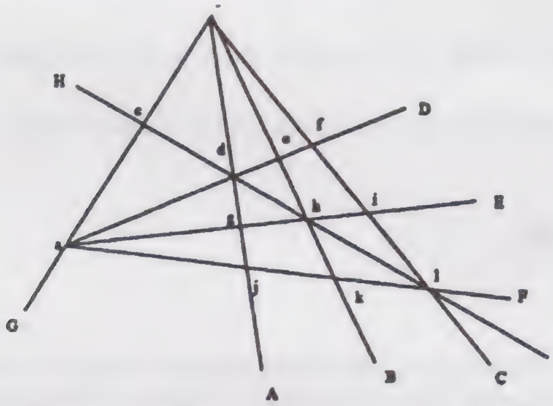


Figure Lines. *Two triplets of lines intersecting in 9 points, three of which are collinear.*

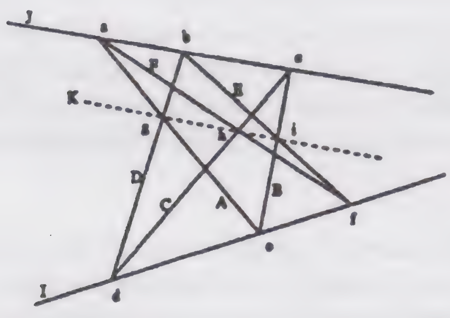
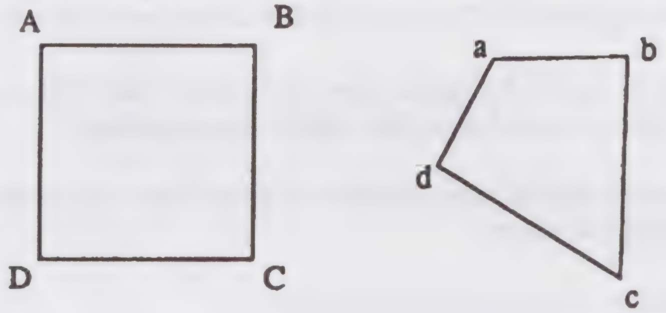


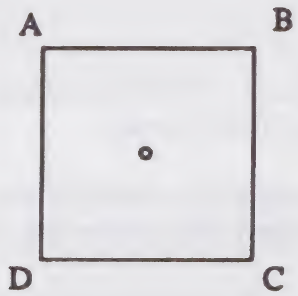
Figure Pappus. *The Pappus configuration, a hexagon whose vertices are alternately on two lines – the three intersection points of opposite edges of the hexagon are collinear.*

Projective Constructions

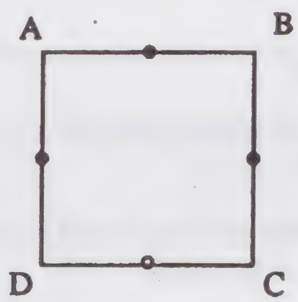
Suppose a square *ABCD* projects to a quadrilateral *abcd*.



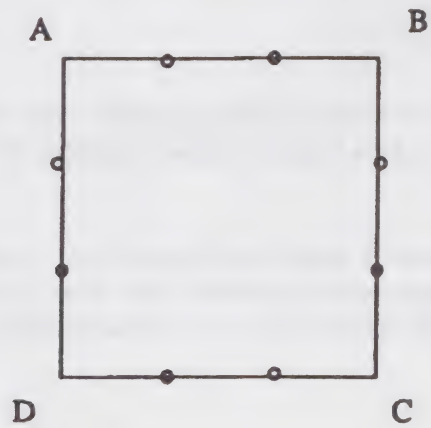
1. How do you find the image of the center?



2. How do you find the images of the midpoints of the edges?



3. How do you find the images of the trisection points of the edges?



39. Duality and the Imagination
17 April 1990

1. What is the dual of a square pyramid? What is the dual of a triangular prism?

2. My bathroom wall is tiled by regular hexagons; what is the dual tiling? The floor is tiled with regular hexagons alternating with equilateral triangles; what is the dual tiling?

3. Can you always color the vertices of a planar graph red, green, blue, and yellow so that no two adjacent vertices have the same color?

4. Why is the Euler characteristic of a 3-manifold always 0?

40. Spherical Triangles

Four "lines" (that is, geodesics or great circles) on a sphere divide the sphere into eight triangles. The triangles group into four pairs of congruent triangles.

1. Write down the inequalities about the angles a, b, c of a spherical triangle which come from the fact that each of these eight triangles must have area greater than 0.

2. The set of possible triples of angles for a spherical triangle forms some shape within the cube $[0, \pi] \times [0, \pi] \times [0, \pi]$. Describe and sketch the shape S determined by the inequalities on areas. Can you explain why S has the symmetry it does?

3. Is every triple of angles within S the triple of angles of a spherical triangle?

4. What shape E within the cube of triples of angles describes the possible triples of angles of a Euclidean triangle?

5. What happens to a spherical triangle when the angles tend toward the boundary of S? (This may depend where on the boundary they are heading).

41. Configuration Spaces
19 April 1990

1. What is the configuration space of the Tower of Hanoi puzzle when there are two disks? When there are three disks? How many moves does it take to move the stack from one post to another when there are 7 disks?

2. One nickel is glued to the table; a second nickel touches the first. The second nickel can slide around as much as it wants so long as it keeps touching the first one. What is the configuration space? What curve is traced out in configuration space when you roll the second nickel around the first one?

42. The Hyperbolic Plane
Assignment Due 24 April 1990

Read Chapters 10, 11, and 12 of *The Shape of Space*. Construct honest-to-goodness hyperbolic paper (not the pale imitation described in the reading).

Geometry Fair!
Come One Come All!

The Geometry fair, featuring final projects from Mathematics 199, will take place this afternoon (May 15), from 2:30 on, in PL (the thirteenth floor of Fine Hall). Everyone is invited.

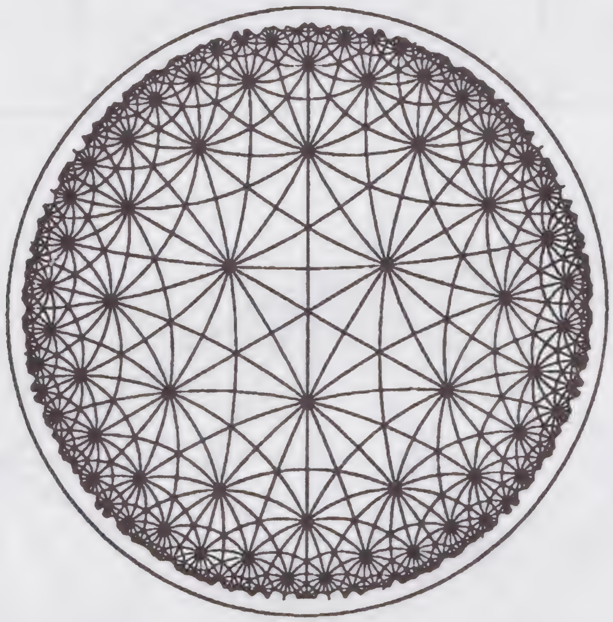
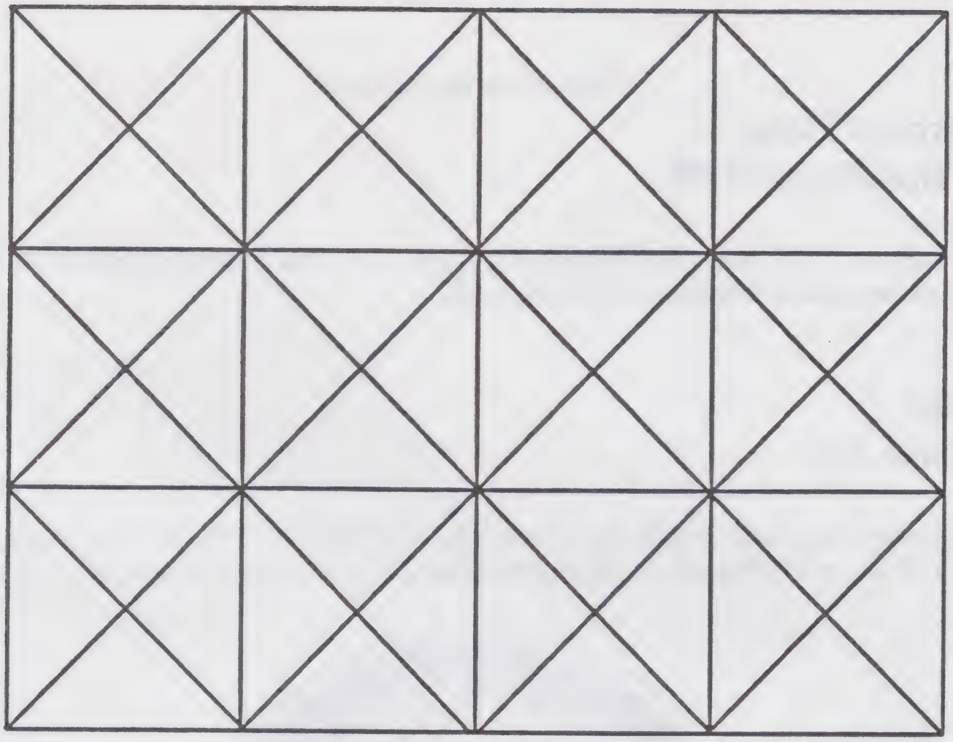
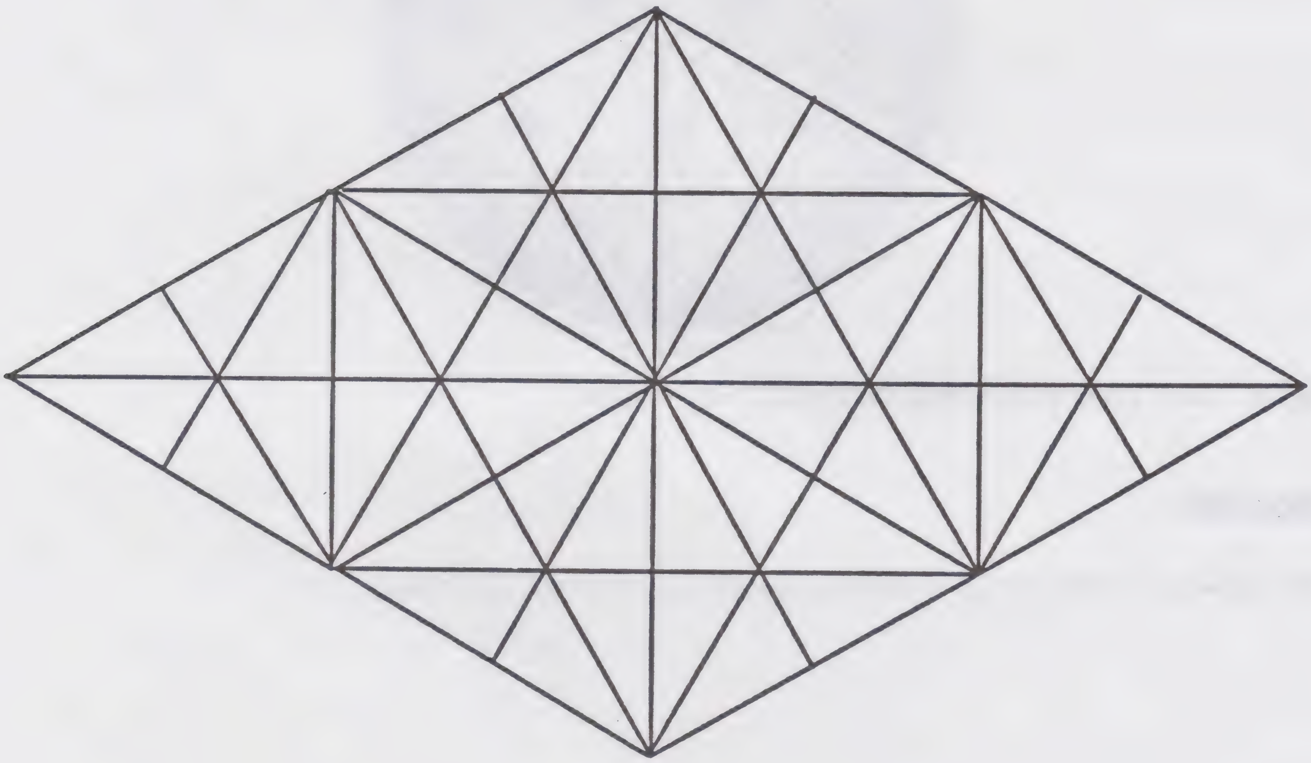


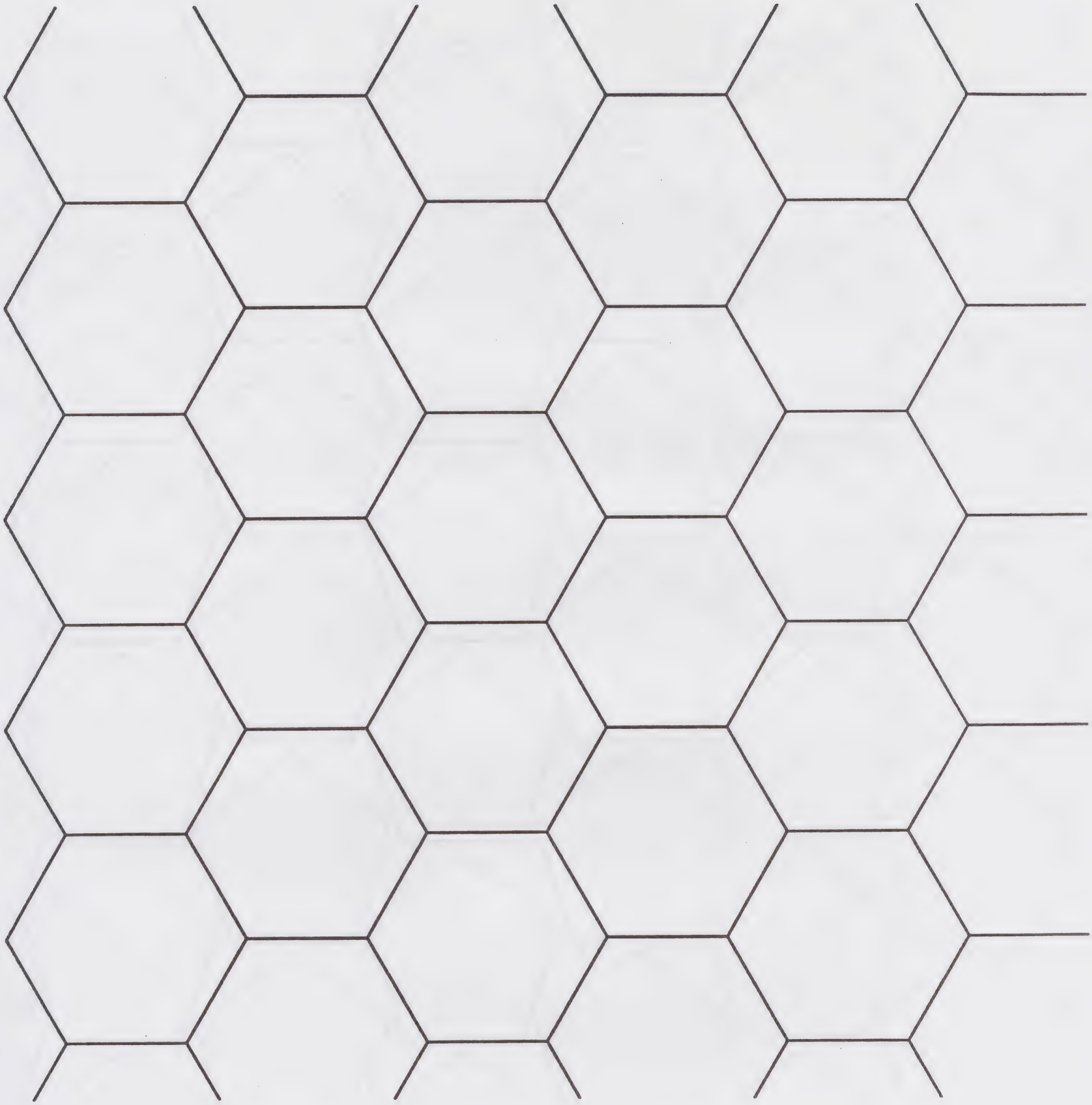
Figure Tiling. *Tiling of Hyperbolic Plane by (2, 3, 7) Triangles.*

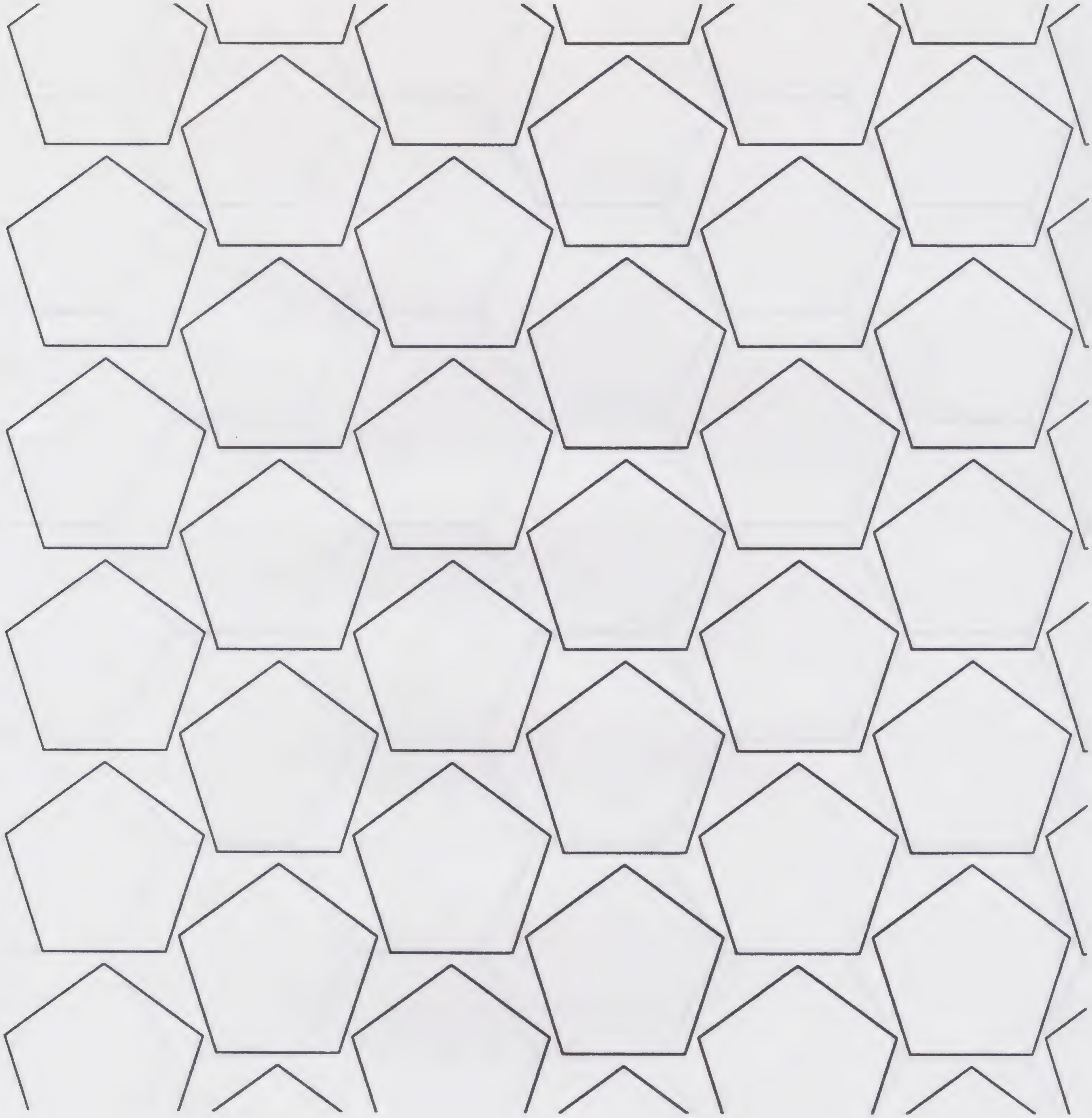
Appendix

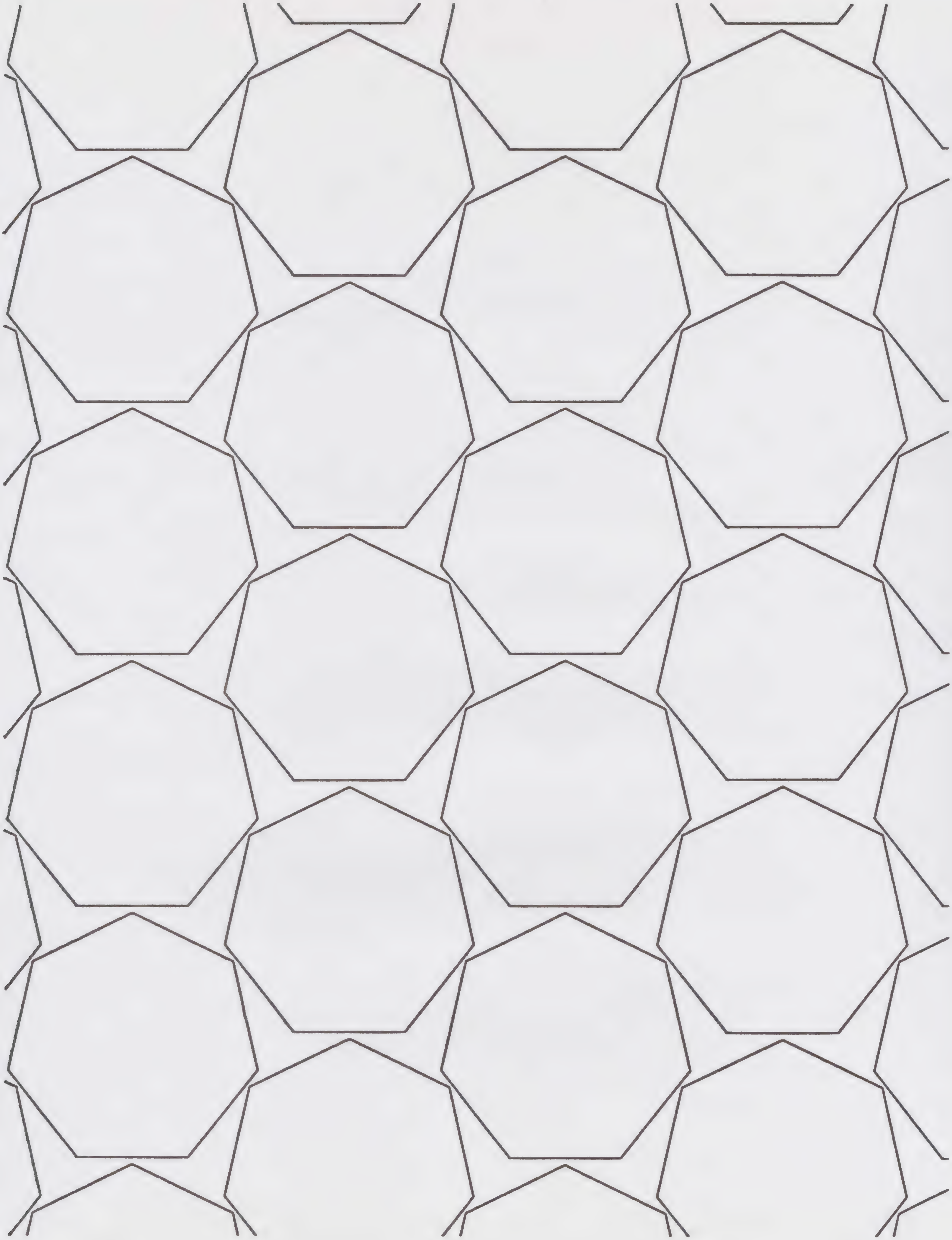
The following templates may be useful in carrying out the above activities.











The Needs of High School Teachers

A Possible Geometry Course for Teachers, and Others

A Personal View

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Prologue

At the University of Wisconsin (Madison) there has been, for at least my 30-year memory, a one-semester Junior-level course in geometry. Although the percentage of prospective high school teachers among the students in this course has varied from 20% to 75%, it has always been thought of as a course for high school teachers. It is a required course for those prospective teachers who major in mathematics.

At the time I inherited this course (in virtue of being the only available person with a substantial background in classical geometry), it was usually taught from books with titles like *Modern Geometry* or A *Modern Introduction to Geometries* or perhaps *Basic Concepts of Geometry*. These books, which I will call "second generation" for reference purposes, tried to replace the "first generation" books of 50 years ago, whose titles were simply *College Geometry*. For these first generation books, the analogy with present day algebra texts is clear: The usual two years of high school algebra are "beginning" and "intermediate" algebra, and the university textbook, which takes up where these leave off and continues in much the same spirit, is "College Algebra." In the same way "College Geometry," took up, and continued in the same spirit, where high school geometry – usually the contents of the early books of Euclid – left off.

For the second generation successors of "College Geometry" there is no parallel with algebra, but a conjectured origin for these books can be given: Those who were graduate students in the 1950s will remember the excitement of discovery involved in books like Hilbert's *Foundations of Geometry*, Landau's *Grundlagen der Analysis* and its fatter child, *The Anatomy of Mathematics* by Kerschner and Wilcox. This was the heyday of logic and foundations, made even more exciting by the doubts cast by Godel's results. At that time, there were even undergraduate courses with titles like "Foundations of Mathematics," which then referred to logic and philosophy, not the mechanical "basics" generally referred to by such titles today. Apparently the second generation geometry books emerged, much like S.M.S.G. and the "new math," from the enthusiasm of those who were particularly impressed by the ferment in logic and foundations of mathematics of the time. These "fundamentalists" had enormous influence in pedagogical circles (which of course especially included geometry because of its old crucial role in the high school curriculum) and it is not unnatural that these second generation geometry books focused on axiomatics, careful proofs, and the relation between "geometries" in the spirit of Klein's Erlanger Programm.

As geometry texts, these second generation "foundations and axiomatics" books were inadequate. They could as well have had algebra or analysis as their subject matter. When these, and other good texts, became unavailable, we were forced to deal with the problem of finding a suitable replacement. So far as I recall, my view at that time (1980) of what should be in a geometry course to meet the combined needs of prospective high school teachers and general mathematics students was the following:

1. The course should contain interesting geometric results – broadly interpreted. (These should be interesting not only to the students but also to the instructor.) These could include some of the more spectacular 19th century-type results, such as Morley's Theorem, the nine-point circle, and the Bolyai-Gerwien theorem on equidecomposability of polygons of equal area, as well as newer results such as those on isometries (e.g., the three-reflection theorem and classification of plane isometries) or convexity.
2. The course should contain useful material for direct use by high school classroom teachers. This means it should be a "refresher" for standard congruence and similarity theorems, circle theorems, and straightedge and compass constructions.
3. Some attempt should be made to show the applications of geometry both in concrete form, such as the use of curves of constant width in the Wankel engine, as well as conceptually, in such contexts as linear algebra, coding theory, finite geometries, and their statistical applications.
4. The course should at least mention the importance of geometry in the history of scientific and philosophical thought. To my mind, this has two main aspects:
   (i) Euclid was the inventor of the axiomatic method which has undeniable – if perhaps exaggerated – importance in modern mathematics, and,
   (ii) The work of Bolyai and Lobachevsky appeared to refute Kant's idea that the human mind was constructed so that only one kind of geometry (Euclidean) was compatible with its workings. The subsequent emphasis on consistency and the eventual evidence that Euclidean and non-Euclidean geometries were equally consistent paved the way for the geometry of Einstein's special relativity as well as Cohen's treatment of the continuum hypothesis.

From this background, I propose to describe the particular course put together some eight years ago and followed since then with little modification. After that description, there are some comments about possible improvements for the course, and a brief indication of the ways it meets some recently enunciated suggestions (Ivan Niven 1987, and MAA Committee on the Mathematical Education of Teachers 1989) for such a course.

Brief Description

The course falls into three main parts:

I. Some Euclidean-type theorems, followed by a quick recapitulation of Euclid's methods and theorems, and a description of the beginnings of non-Euclidean (hyperbolic) geometry.

II. The geometry and algebra of plane isometries, with applications to the study of repeated patterns in art and archaeology.

III. An introduction to hyperbolic geometry and the Poincaré model, concluding with an application to the study of Escher's Circle Limit prints.

For variety, a two-week module, more or less independent of the rest of the course, is inserted on one of the topics: equidecomposability of polygons and polyhedra; Euler's polyhedron formula and its applications; the Fibonacci (and related) sequences and their geometrical and botanical manifestations; higher dimensional polytopes, especially cubes and their duals.

Details

Part I of this course was suggested especially by the first chapter of David Kay's book, and by the first chapter of Coxeter-Greitzer's *Geometry Revisited*. Kay's idea was that although an axiomatic treatment might still be appropriate for geometry, the student does not want to begin in this way. Thus, he introduced a number of attractive results at the beginning, accompanied by the proofs that most of us would recognize as proofs. Among these are the nine-point circle theorem for triangles, Morley's theorem on angle trisectors, and some properties of the Fibonacci sequence and the golden ratio.

My own choice was to follow the more systematic arrangement of Coxeter-Greitzer, so that the "attractive results" appear in the context of two main strands, rather than as completely isolated

examples. These are, first, Ceva's theorem, some of its applications and other simple theorems suggested by them; and, second, a variety of applications of the theorem that the angle subtended by a circle arc at its center is twice the angle subtended at a point on the circle outside the arc.

The proofs of these theorems serve to remind the students of some of the standard facts of Euclidean geometry, such as the properties of perpendicular bisectors and angle bisectors, conditions for congruence and similarity of triangles, angle sums in polygons, etc. Then Part I concludes with a Proposition by Proposition survey of Euclid's Book I, pointing out along the way that Euclid proved Propositions 1-28 without making use of the parallel postulate. In this context, the structure of Euclid's system is emphasized, both as the prototype of an axiomatic system and, more particularly, as a collection of theorems of absolute geometry, independent of choice between the hyperbolic and Euclidean parallel axioms. Thus, a ready-made body of theorems (especially the "weak exterior angle theorem," Prop. 16) becomes available for use in Part III.

Part II begins with the simple proof of the three-reflection theorem, and the consequent classification of plane isometries. Some applications of the algebra of isometries, such as determining the center and angle of the rotation which is the product of two rotations, are given. It is pointed out that some results are more easily obtained in this way, some are more easily obtained in Euclid's framework, while still others (such as the product of translations) have vectors as their natural habitat.

The classification of isometries leads to crystallographic classification of patterns according to their isometries. The enumeration of the seven frieze patterns is done completely, and some hints (such as the crystallographic restriction) are given toward the enumeration of the 17 plane patterns. This machinery is then applied to the analysis of real-world patterns from various sources. Many, though not all, students have commented that this is a particularly attractive part of the course, since they had never realized that geometry could be applied in this way.

At this point, a short, unrelated section, as mentioned above, is usually treated as a sort of intermission entertainment before the final Part III. However, the equidecomposability theorems give an opportunity to apply the information about various subgroups of the full isometry group that appeared already in Part II.

Part III gives a short introduction to hyperbolic geometry following Lobachevsky, since his monograph fits in well with the Euclidean style of Part I. A careful path is chosen so that few proofs have to be omitted in order to get some of the essential flavor of hyperbolic geometry, including especially the role of the angle sum theorem for triangles. Emphasis is always placed on the role of Euclidean geometry as "intermediate" between spherical and hyperbolic geometry. In this way, some spherical geometry is informally taught. (It is a continual surprise that many students are not yet aware that the shortest route from New York to London is <u>not</u> along a circle of latitude.) The Poincaré model is introduced. Unfortunately, most students have not understood the isometry theorems well enough to fully grasp the significance of inversions as the analog of reflections. Hence, some of the interesting properties of the Poincaré model must be stated without proof.

This part concludes with the detailed study (following Coxeter) of Escher's four Circle Limit prints as representations of various features of the hyperbolic plane, especially tessellations. In this context, regular tessellations of the sphere and Euclidean plane are enumerated, concepts of truncation, duality, and semi-regularity are introduced, and regular compound tessellations of all three planes are also discussed. A set of overlays for Escher's prints is given to the class to aid in this analysis, and other visual supplements come from a short film on Escher and Douglas Dunham's computer drawings in Escher style. When time permits, the Apple computer version of Dunham's program is made available for student experimentation. The relative unfamiliarity of current students with the work of Escher is partially compensated for by their greater familiarity with computers.

Improvements and Comments

(i) For decades, complaints have been made saying that the traditional geometry courses in high school concentrate on plane geometry, while neglecting the geometry of space. Anyone who teaches a linear algebra course finds that such remarkable facts as "In R^4, two planes may have only a single point in common" often fall on the deaf ears of those who think the same is true in R^3. The course described here does nothing to address this complaint.

Possible remedies might be to include some interesting space results in Part I and/or to include a survey of Euclid's Book XI. In the attractive book of Melzak (1983), there are chapters on "Oblique sections of certain solids" and "Simple geometry and trigonometry on the sphere," which give an idea of what could be attempted in this direction. The simple derivation of the area of a spherical triangle in terms of its "spherical excess" provides motivation for the corresponding hyperbolic theorem on "angular defect" (stated without proof). In any case, this derivation deserves to be better known.

In Part II, the study of isometries could usefully be extended to the classification of space isometries, since the basic tool, the four-reflection theorem, is easily understood. However, a detailed study of the crystallographic space groups is too intricate and specialized. This objection applies even more strongly to the natural extension of the equidecomposability theorems to Dehn's theorem on the non-equivalence of the cube with the regular tetrahedron.

At one time, a one-semester course in "solid geometry" was offered in many high schools. Perhaps it should be reinstated, instead of calculus.

(ii) Occasionally, students want to hear more about certain topics to which they have heard reference. These still include angle trisection and the general topic of impossibility proofs, as well as modern topics such as fractals. (Surprisingly little has so far come to their attention about Penrose tiles and quasi-crystals.) The traditional questions can be dealt with as they arrive, but I have not yet found a reasonable way to include fractals.

(iii) Since this course is aimed at prospective high school teachers, it is sometimes perceived as too shallow or too slow by more "serious" mathematics students. In fact, it is likely that these latter students do not register for the course. It is not easy to see how to design a single course to meet the needs of both "terminal" students such as prospective high school teachers and students who may go on to further work in topology or differential or algebraic geometry. Twenty years ago, this was not such a problem, since at that time we still regularly taught more specialized courses such as projective geometry and convexity, which could be taken by students whose taste for geometry had been whetted by the present course. To a certain extent, students who might have taken those courses now take combinatorics, or more distantly related courses such as coding theory, linear programming, or numerical analysis.

Suggestions of Niven (1987) and MAA (1989)

(i) The recent *NCTM Yearbook*, "Learning and Teaching Geometry, K-12," (1987), is less useful than might be hoped as a guide for designing a suitable college geometry course for high school teachers. However, in the yearbook, Ivan Niven presents a list of nine specific recommendations with which the present course can be compared. These are, with some paraphrasing:

1. Teach beginning geometry in the same way as beginning algebra or calculus, without excessive emphasis on rigor.
2. Get to the heart of geometry as soon as possible.
3. Use algebra and analytic geometry, as well as Euclidean methods.
4. Use diagrams in all explanations.
5. Relate geometry to the mainstream of mathematics and the real physical world.
6. Avoid elaboration of the obvious.
7. Omit proofs of some difficult theorems.
8. Include many intermediate level (in contrast to easy) problems.
(9. Tell students the full story about the trisection of the angle.)

Aside from 9., which appears to be a cry of help from someone beset by angle trisectors, these appear to be agreeable recommendations. Except for 3. and 8., this is much the implicit guide for the course described above, though my interpretation of the "real physical world" is probably somewhat different from Niven's.

(ii) The MAA (1989) "Standards" document circulated in connection with this conference recommends the inclusion in the college preparation of teachers of "the axiomatic foundations of Euclidean geometry, principles and examples of non-Euclidean geometries, principles of transformation geometry, and applications of geometry to science and technology as well as the arts." The teachers should "develop conceptual understanding of the role of congruence and similarity in the classification of geometric figures. When possible, mathematics teachers' geometry training should include work with state-of-the-art technology for the exploration of geometric concepts." These seem to be very modest goals, which come close to being met by the course described above. Are they in fact too modest?

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Investigating Volumes: The Air France Cup

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Geometry is about phenomena. More than in any other branch of mathematics, geometry can be seen and felt. But it is more than the experience of geometric form that characterizes the subject – we have a natural process of abstraction and measurement that we encounter first on an intuitive level and then gradually refine throughout our mathematical education. That progressive aspect cannot be stressed too much, nor can the interconnections between geometric forms and other parts of mathematics. In my opinion, we often tend to withhold concepts from students until we feel that they are ready for them. Our attitudes are quite justified when the concept is extremely abstract, like some of the constructions in algebra or measure theory, but they can be mistaken when it comes to the presence of geometry in early education. One of my favorite illustrations of the abuse of this treatment of geometry has to do with the volume of a cone. For many students, the first time they see the formula for the cone's volume is the same day that they calculate this volume in a calculus course, as an example of a volume of revolution. At least, that is the way it always seems when I present the topic to calculus students. There should be an "aha!" from the class as a whole, when all at once there appears on the blackboard a justification for a rule that had been learned years earlier, and experienced years before that. Instead, the cone volume is submerged in a collection of frequently artificial stock problems at the end of a section on applications of integration.

Even more distressing is the fact that some students will remember a formula for the volume of a cone but have no idea about the volume of a pyramid. (After all, a pyramid is not a surface of revolution, so it only shows up in an even more optional section on applications, for volumes with known cross-section.) Worse still is the fact that many students do not seem to appreciate that the two formulas are related – the one-third that appears as a coefficient in each case might almost be viewed as a coincidence.

In my essay on *Dimensions* in the collection *On the Shoulders of Giants*, I began with the work of Friedrich Froebel, who championed this experimental approach to geometry in the early part of the nineteenth century. He concentrated on pre-school, and presented shapes to his pupils for them to play with. It was directed play to be sure, led along fairly well-determined paths by well-prepared Kindergarten teachers, but still the spirit of play was preserved. Pre-schoolers were not expected to prove theorems. But they could be expected to recognize that four square tiles fill a tray with edge length twice that of a small square, and eight cubes fit in a critical box with edge length twice that of a small cube. Moreover, they recognized that a similar relationship held for nine tiles or 27 small cubes when edge lengths tripled. Later on, they would recognize the patterns in the exponents that lead to content formulas in all dimensions, and they would say "aha!" (or something equally appreciative).

The block play of Froebel's students led to other decomposition theorems as well, so for example, they would not be surprised to see that a triangle has half the area of an associated parallelogram, or even that a pyramid has one-third the volume of an associated prism. That perhaps is a key to some of the confusion that some students experience later on – the cone is viewed as something quite separate from anything else. It doesn't seem to have the same sort of relationship with anything two-dimensional that occurs between a pyramid and a triangle. The fundamental fact is that cones don't pack – no matter how we arrange them in a box, there is space left over. That isn't the case for triangles of course – they can

be repeated so that they fill the entire plane with no space remaining. The comparable figure in three-space is not a cone but a pyramid (a cone over a polygonal region). Not every pyramid can be used to fill space, but some can, and in particular it is possible to divide up a cube into three congruent pyramids. I think of that as being a fundamental insight, something that every person should know. There are not so many facts that I put on that same level.

To get from that fact to the general theorem about volumes of pyramids requires a stretching of the case for a pyramid in a cube, a variable stretching in different directions. If we double the length in a given direction, then the volume doubles. That's another fundamental insight. Such concepts should be familiar long before they become formalized in the calculus sequence.

Perhaps what I would suggest at the very least is that we incorporate some of these principles into the precalculus course. It isn't enough just to teach people how to factor polynomials, use the quadratic formula, and handle elementary trigonometric and exponential expressions, and to parse standard word problems. We should also give them some appreciation of area and volume so they won't be encountering these notions for the first time only after they are already confused by the concept of limit.

We can then reinforce these fundamental notions in our college geometry courses. An example I have used several different times in classes at quite different levels is the Air France Cup. That airline provides a plastic cup that is circular on the top and square on the bottom. There is a hole in the tray into which one can drop the cup so that it goes down half way. What is the shape of the holes and what is the volume of the cup? The answer to this, like the answer to so many mathematical questions, is, "it depends." It depends on the radius of the top rim, the length of the edges of the bottom square, and the fact that the top and bottom are in parallel planes. The volume also depends on the distance between these parallel planes, although the shape of the middle slice does not, at least if we make a further crucial assumption, namely that the cup is convex, and that it is the smallest convex set containing the top and bottom. And neither the volume nor the shape of the halfway slice depends on whether or not the center of the circle lies directly above the center of the square.

Convexity should be a familiar concept throughout the curriculum, beginning with pre-school tiles and appearing in every course. Still it may take some work for students to figure out the shapes of intermediate slices just based on the three-dimensional convexity assumption. They can be guided, of course, by some limiting cases – if the square has zero side length, then we have a conical Dixie cup, with known volume and cross-sections, and if the circular rim has zero radius, we have a square-based pyramid, again well-understood.

Since it is the curved sides of the figure that may be a source of confusion, it may be helpful to consider a more polyhedral example, with a square on top and another square on the bottom (dubbed the "Square France Cup" by one student). Now there is a simplification on one hand but a complication on the other. If the squares are of the same size and with edges parallel, then the convex set is a prism if the centers lie over each other, and a square-based parallelepiped with the same volume as that of the prism, if the centers are not so situated.

Things are not too much more difficult if the squares have unequal edge lengths. In this case, the convex set is a frustrum of a square-based pyramid and the cross-sections are squares, with edge length changing linearly as we go from the bottom to the top. This problem was solved already nearly four thousand years ago by the Egyptians (for whom the calculation of the volume of an incomplete pyramid was of practical importance as well as theoretical interest). Completing the frustrum for a pyramid leads to a solution of the volume problem, using a bit of elementary algebra (**Figure 1**). This solution also indicates the presence of a mixed term, relating the top and bottom edge lengths. It is not as well known as it should be, despite the efforts of papyrus scribes.

Note that

$\frac{x}{r} = \frac{x+h}{s}$ by similarity. Thus,

$$V = \frac{1}{3}(s^2(x+h)) - \frac{1}{3}(r^2x)$$

$$= \frac{1}{3}(s^2(\frac{sx}{r}) - r^2x)$$

$$= \frac{1}{3}(s^3(\frac{x}{r}) - r^3(\frac{x}{r}))$$

But $sx = rx + hr$, so $x = \frac{rh}{s-r}$, $\frac{x}{r} = \frac{h}{s-r}$

so $V = \frac{1}{3}(h)(\frac{s^3 - r^3}{s-r}) = \frac{h}{3}(s^2 + sr + r^2)$.

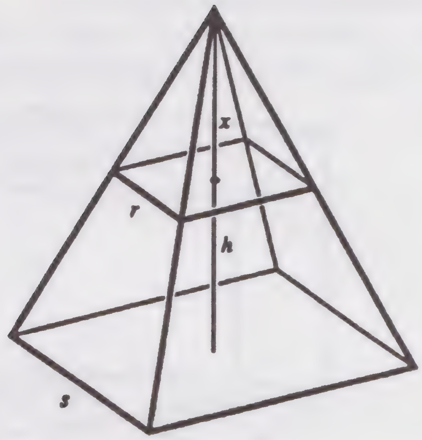

Figure 1.

We recognize the parts that reduce to the volume of a pyramid if $r = 0$ or if $s = 0$, and we have this intermediate part as well. Note that the halfway slice is a square with side $(1/2)(r+s)$, so with area $(1/4)s^2+rs+(1/4)r^2$. This expression contains the term rs which appears in the volume formula for the frustrum of the pyramid and we might expect a relationship between the area of this middle slice and the full volume of the solid.

We can easily use this approach to find the volume of a cup with rectangles of the same shape as its rims, just by applying the stretching principles. Another modification of the formula is necessary, however, if the top rectangle is rotated by a quarter turn. The convex hull is not so hard to determine in this case – it has a boundary consisting of four trapezoids – and the middle slice is a square (**Figure 2**).

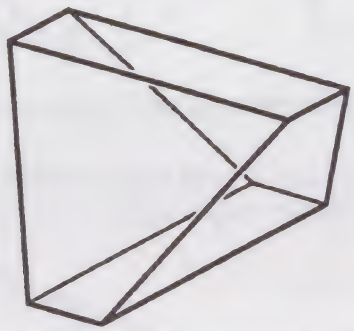

Figure 2.

This example makes it clear that the volume of the convex hull (the smallest convex set containing them) depends on more than just the areas of the top and bottom rims. An extreme case of some importance occurs when the rectangles shrink to segments. If the segments are parallel, we get a rectangle as the convex hull, with zero volume. If the segments are in perpendicular directions, then the convex hull is a tetrahedron. We can divide this into two triangle-based pyramids,

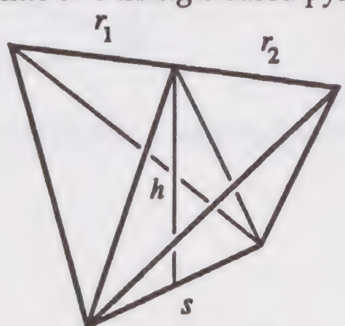

Figure 3.

so we can calculate the volume by knowing the area, $(hs/2)$ so we have $V = 1/3(hs/2)\ (r_1+r_2) = (1/6)\ hsr$. Note that this formula works if r and s are different, just as long as the lines which contain the two segments are perpendicular (**Figure 3**).

We can see this another way by working with subtraction rather than addition. We complete the figure to a prism with parallelogram base (**Figure 4**).

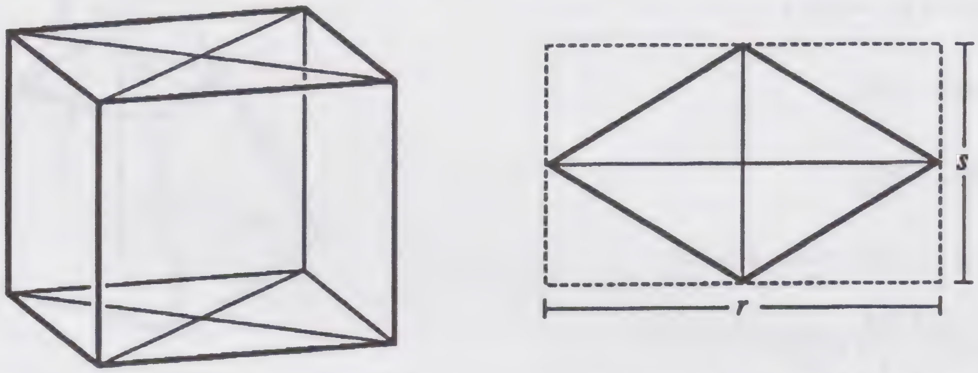

Figure 4.

The area of the parallelogram will be $(rs)/2$, and the tetrahedron remains in the middle after we slice off four corners of the prism.

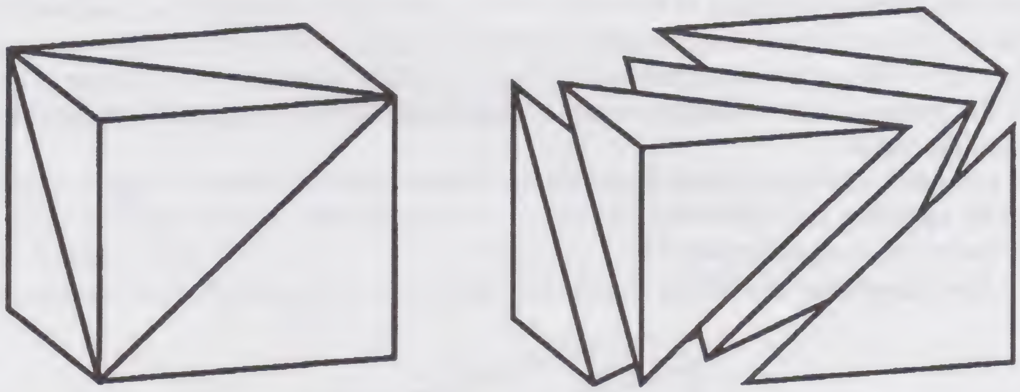

Figure 5.

Each corner slice has volume $(1/3)h((1/4)\ rs)$ so the volume of the inside tetrahedron is $1/2(rs)\ h - 4((1/3)h((1/4)rs)) = (1/6)\ rsh$ (**Figure 5**).

We can use this result to solve the Square France Cup problem when the top square is rotated 45° with respect to the bottom. In this case the convex hull is a square antiprism (**Figure 6**).

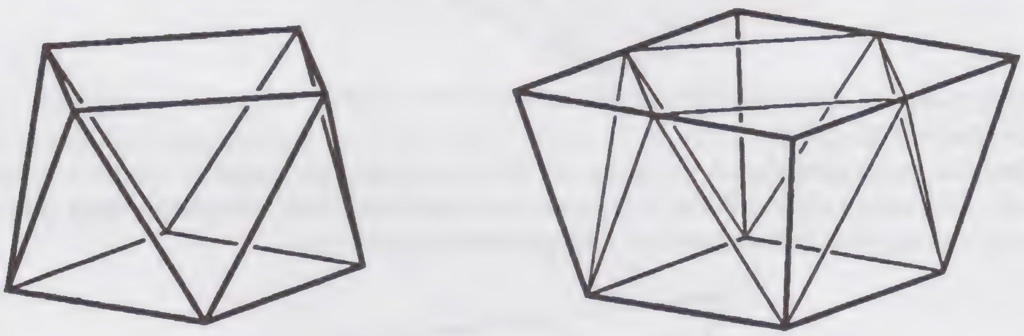

Figure 6.

We may obtain such a figure by removing four triangle-based pyramids from a frustrum of a pyramid.

We can also decompose this figure another way which leads to the solution of the original problem (**Figure 7**).

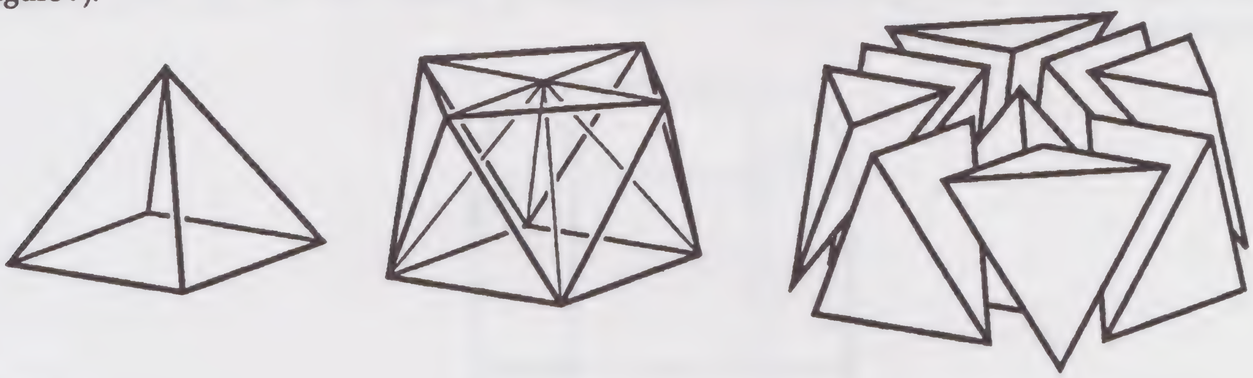

Figure 7.

If we remove from the figure the square-based pyramid with the center of the top square as its apex, we can decompose the remaining figure into the four triangle-based pyramids with their apices at the vertices of the lower square, and four tetrahedra with one edge in each of the two parallel planes, lying in perpendicular directions. The entire volume is then,

$$\frac{1}{3}s^2h + 4\cdot\left(\frac{1}{3}h\right)\left(\frac{r^2}{4}\right) + 4\cdot\frac{1}{6}\cdot h\cdot s\cdot\left(\frac{r}{\sqrt{2}}\right)$$

$$= \frac{1}{3}(h)\,(s^2 + r^2 + \sqrt{2}\,rs).$$

Practically the same decomposition works for the Air France Cup. We note that the planes through the sides of the square base which just touch the circular top will contain triangles lying in the boundary of the convex hull.

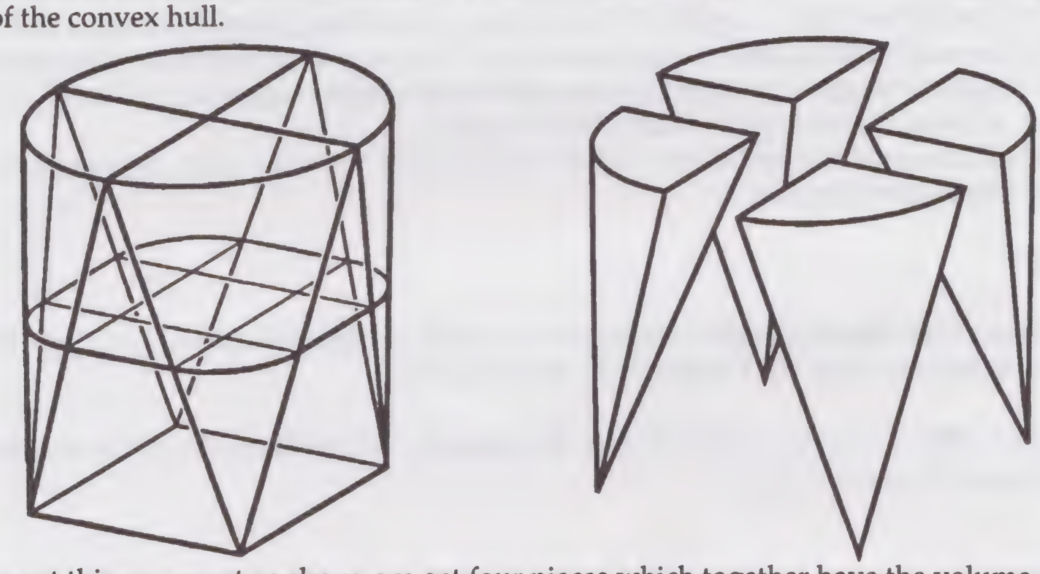

Figure 8.

In fact, if we cut this cup apart as above, we get four pieces which together have the volume of a circular cone, and the remaining pieces are the same in the Square France Cup. Thus, the volume is $(1/3)s^2h + (\pi/3)t^2h + 4(1/6)(tsh) = (1/3)h(s^2 + \pi t^2 + 2ts)$ (**Figure 8**).

In the previous diagrams we can also see the answer to the shape of the halfway slice.

We have four quarter-circles each with radius $t/2$, contributing a total area $\pi(t/2)^2$, together with a square of side length $s/2$ and four rectangles of side lengths $s/2$, and $t/2$. Thus, the area of the middle slice is $\pi t^2/4 + s^2/4 + st$ (**Figure 9**).

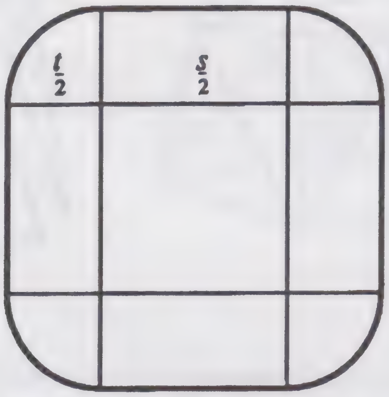

Figure 9.

So, after a long process, we have solved one particular geometric problem. In that process, we have used any number of important geometric techniques, and it is possible to continue these ideas in several different directions, to study formulae for different shapes of the top and bottom for example, or to generalize to cases where the planes of the two figures are not parallel. It is also possible to consider generalizations of this result to higher dimensions! The methods used up to this point have been almost entirely synthetic, and indeed they could be used already in secondary school or lower. A small bit of trigonometry makes it possible to extend the result for a tetrahedron to the case where the opposite edges lie in lines with directions making an angle θ (leading to the formula $1/6\, hrs \sin \theta$). We can introduce coordinates, and begin to write volume formulae in terms of coordinates, again with generalizations for higher-dimensional space.

The Air France Cup is a good example of an easily stated problem that leads to investigate a great many different geometric topics. The final formula is not so important as the process by which students learn to recognize various aspects of the problem, formulating related problems and seeing how the solution of a special case can lead to a general method. The process of generalization appears to be quite natural. Students can grow to appreciate the way mathematicians approach geometric phenomena, and that is one of the greatest insights we can hope to convey.

The illustrations for this article were rendered by Davide Cervone, using the program Aldus Freehand on a Macintosh computer.

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Geometry in Utopia

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A well-known parable relates how an elephant is described to a blind man. One man verbalizes a detailed description of the trunk, another man describes the ear, and a third person describes the foot. From these local descriptions one tries to get a global view of the beast.

Geometry makes up a vast part of mathematics. The purpose of this note is not to propose that all the ways of viewing geometry are equally important or that all its parts deserve a place in the curriculum. Rather, my purpose is to show the richness of geometry and to urge that more of its parts deserve study and attention in our schools.

Listed below are a variety of topics involving geometry. The section heads are somewhat arbitrary and appear in random order. Many of the topics overlap heavily, yet despite this, it seems convenient to list them all separately, so as to show the many different perspectives via which geometric ideas can be approached. At the college level, these topics can be used to put together a survey course or be used as topics for talks at a mathematics club. In many instances, they can serve as topics in courses not specifically about geometry. At the high school level, these topics can be included in existing courses or can be used as topics for mathematics club talks or research seminar investigation. Many topics in geometry can be studied without large amounts of prior mathematical training. Many areas of geometry also have unsolved problems that can be simply stated and require relatively little formal study to work on them. This allows an average student to see that there are many mathematical facts still to be discovered. Also, problems of this kind are suitable for undergraduate and high school research projects.

Geometry has always found many applications within mathematics. Recently, however, geometric ideas are being applied to biology, computer science, engineering, and many other areas. These applications can make clear to laymen how mathematics is affecting their lives.

In thinking about the topics below, I feel it is useful to distinguish between what I will call "geometry" and "visual reasoning." For each of the sixty areas, "geometry" of the area refers to the facts that make up the subject. "Visual reasoning" refers to the value of drawing diagrams and making models as a mode of thinking and understanding. Of course, "visual reasoning" can be used to obtain insight into areas of mathematics other than geometry, for example algebra. On the other hand, many geometric facts are most easily demonstrated by methods that are not geometric.

1. Axiomatics

a. Euclidean geometry
b. Bolyai-Lobachevsky geometry
c. Projective geometry
d. Concepts of independence, completeness, and categoricalness

2. Finite Geometries

a. Finite affine planes
b. Finite projective planes
c. Finite hyperbolic planes
d. Coordinate systems for finite planes
e. Block designs
f. Applications to statistics and error correcting codes

3. Geometric Transformations

a. Rigid motions and isometries
b. Similarity transformations
c. Inversions
d. Affine maps
e. The Erlangen Program
f. Applications to computer graphics

4. Symmetry

a. Analysis of designs using group theory
b. Frieze and wallpaper groups
c. Applications in archaeology etc.
d. Fabrics
e. Color symmetry
f. Local symmetry

5. Tiling Problems

a. Tilings with regular polygons
b. Tilings with convex m-gons
c. Polyominoes
d. Penrose tiles (aperiodic tilings)

6. Lattice Point Problems

a. Pick's Theorem
b. Minkowski's Theorem
c. Integer programming

7. Graph Theory

a. Euler Circuits (Chinese Postman Problem)
b. Hamiltonian Circuits (Traveling Salesman Problem)
c. Trees
d. Euler's Formula
e. Applications in Operations Research

8. Taxicab Geometry (Minkowski planes)

a. Abstract distance functions
b. Conics

9. Convexity

a. Curves of Constant Width
b. Helly's Theorem
c. Support Lines
d. Rotors

10. Discrete Geometry

a. Distances generated by point sets
b. Coloring points
c. Sylvestor's problem

11. Geometry of Surfaces

a. Orientability
b. Moebius strip and Klein bottle
c. Spheres with handles
d. Embedding graphs on surfaces

12. Polyhedra

a. Classification
b. Existence
c. Euler's polyhedral formula
d. Steinitz and Eberhard's Theorems
e. Geodesic Domes
f. Polyhedra in higher dimensions

13. Equidecomposability (Dissection Problems)

a. Bolyai – Gerwin Theorem
b. Dehn – Hadwiger Theory
c. Banach – Tarski Paradox

14. Differential Geometry

a. Curvature
b. Geodesics
c. Soap bubbles

15. Computational Geometry

a. Convex hulls
b. Point location
c. Triangulation
d. Visibility
e. Voronoi diagrams

16. Packing and Covering Problems

a. Circle and sphere packings
b. Circle and sphere coverings
c. Box packings

17. Rigidity of Structures

a. Rigidity of polyhedra
b. Rigidity of graphs
c. Rod and cable structures

18. Geometric probability

a. Buffon needle problem
b. Alternative probability models

19. Digital geometry

a. Pixel representation of objects
b. Connectivity
c. Convexity analogues
d. Shape recognition
e. Contour operations

20. Knots

a. Graphs of knots
b. Classification
c. Knot invariants
d. Applications of physics

21. Isoperimetric Problems

a. Area versus perimeter
b. Volume versus surface area
c. Geometric inequalities
d. Isoperimetric inequalities

22. Cartography

a. Stereographic projection
b. Equiareal projection
c. Azimuthal equidistant projection
d. Mercator projection

23. Geometric Extremal Problems

a. Fagnano's problem
b. Steiner symmetrization
c. Steiner-Fermat point
d. Steiner trees

24. Geometric Games

a. Pursuit curves
b. Graph games
c. Connect dots and box games
d. Conway's Life

25. Plane Curves

a. Cardiods
b. Conchoids
c. Lemniscates
d. Evolutes

26. Distances

a. Abstract distance
b. Euclidean distance
c. Taxicab distance
d. Spherical distance
e. Geodesics

27. Coordinate Systems

a. Rectangular coordinates
b. Polar coordinates
c. Barycentric coordinates
d. Parallel axis coordinates

28. Geodesy

a. Triangulation schemes
b. Orbit calculations
c. Celestial mechanics

29. Algebraic Geometry

a. Genus of a curve
b. Intersections of polynomial functions
c. Bezout's theorem
d. Groebner bases
e. Splines

30. Groups

a. Groups of transformations
b. Polyhedral group theory
c. Symmetry groups of polyhedra, tilings, and patterns
d. Escher patterns

31. Linear Programming

a. Polyhedral cones
b. Assignment and transportation polytopes
c. Path algorithms

32. Conic Sections

a. Parabola
b. Ellipse
c. Hyperbola
d. Degenerate conics
e. Applications

33. Polygons

a. Convex polygons
b. Simple plane polygons
c. Non-plane polygons

34. Arrangements of hyperplanes

a. Cell structure of lines in the Euclidean and projective planes
b. Cell structure of planes in Euclidean 3-space
c. Simplical arrangements of lines
d. Cell structures of arrangements of pseudolines

35. *N*-dimensional geometry

a. Regular polytopes
b. Spheres: volume and surface area
c. Hypercubes

36. Space-time geometry

a. Special relativity
b. General relativity

37. Display of Data

a. Histograms, pie charts, frequency polygons, ogives
b. Chernoff faces
c. Multi-dimensional display methods

38. Image Processing

a. Chain codes
b. Image compression
c. Registration

39. Linkages

a. Peaucellier's linkage
b. Hart's linkage
c. Planimeters
d. Kempe's theorem
e. Applications to robots

40. Computer Vision

a. Motion planning for robots
b. Edge detection
c. Motion detection
d. Euler operators
e. Interpretation of line drawings

41. Fractal Geometry

a. Hausdorf dimension
b. Self-similarity
c. Space filling curves

42. Geometric Constructions

a. Constructions with compass alone
b. Constructions with ruler alone
c. Constructions with collapsing compass

43. Geometric Puzzles

a. Dissection puzzles
b. Rubik's cube
c. Box packings

44. The Pythagorean Theorem

a. Algebraic proofs
b. Geometric proofs
c. Generalizations

45. Geometric Inequalities

a. Arithmetic and geometric means
b. Isoperimetric inequalities
c. Triangle inequalities

46. Integral Geometry

47. Optical illusions and Moire Patterns

48. Inversive Geometry

a. Orthogonal circles
b. Coaxial circles
c. Moebus planes
d. Circle-preserving transformations

49. Solid Modeling

a. Representation of Surfaces
b. Data structures
c. Euler operators
d. Splines

50. VLSI Design

a. Routing problems
b. Embedding graphs in a mesh (grid)
c. The Shuffle-exchange graph

51. Error Coding Codes

a. Hamming distance
b. Codes and sphere packings
c. Codes and finite geometries
d. Applications to digital technologies

52. Cellular Automata

a. Self-reproduction
b. Garden of Eden configurations
c. Conway's Life

53. Robotics

a. Vision systems for robots
b. Global motions planning
c. Local motions planning

54. Crystallography

a. Crystal lattices
b. Crystal growth
c. Space fillers

55. Dynamical Systems

a. Iteration of functions
b. Orbits
c. Chaos

56. Shape Grammars

57. Catastrophe Theory

58. Geometry and Complex Numbers

59. Paper folding and Origami

60. History

a. Egyptian geometry
b. Greek and Roman geometry
c. Geometry in the Renaissance
d. 17th century geometry
e. 18th century geometry
f. 19th century geometry
g. 20th century geometry

0. GENERAL

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